The Bright Side of Mathematics

Start Learning Reals – Part 3

complete number line $\mathbb{R}$

Axioms of the reals: A non-empty set $\mathbb{R}$ together with operations $+ , \cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x\cdot(y+z) = x\cdot y + x\cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $|x| := \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$

Important facts: There is a set with all these properties (Existence) (Construction) see next video

and it is uniquely determined by these properties. (Uniqueness) (Identification/ Isomorphism)

Show: For all $x \in \mathbb{R}$, we have: $0 \cdot x = 0$ (*) (by only using the axioms).

Proof:

$$0 \overset{(A)}{\underset{\text{inverse}}{=}} (0\cdot x) + (-0\cdot x) \overset{(A)}{\underset{\text{neutral}}{=}} ((0+0)\cdot x) + (-0\cdot x)$$

$$\overset{(D)}{=} (0\cdot x + 0\cdot x) + (-0\cdot x)$$

$$\overset{(A)}{\underset{\text{associativity}}{=}} 0\cdot x + (0\cdot x + (-0\cdot x)) \overset{(A)}{\underset{\text{inverse}}{=}} 0\cdot x + 0 \overset{(A)}{\underset{\text{neutral}}{=}} 0\cdot x$$

Show: For all $x \in \mathbb{R}$, we have: $(-1)\cdot x = -x$ (by only using the axioms).

Proof:

$$-x \overset{(A)}{\underset{\text{neutral}}{=}} 0 + (-x) \overset{(*)}{=} 0\cdot x + (-x) \overset{(A)}{\underset{\text{inverse}}{=}} ((-1)+1)\cdot x + (-x)$$

$$\overset{(D)}{=} (-1)\cdot x + 1\cdot x + (-x) \overset{(A),(M)}{=} (-1)\cdot x + 0 \overset{(A)}{\underset{\text{neutral}}{=}} (-1)\cdot x$$