The Bright Side of Mathematics

Start Learning Numbers – Part 7

In $\mathbb{N}_0$ $\quad 4 + x = 0 \quad$ is not solvable! No "inverse" of 4.

$$\mathbb{Z} := \left\{ [(a,b)]_\sim \;\middle|\; (a,b) \in \mathbb{N}_0^2 \right\} =: \mathbb{N}_0^2 / \sim$$

with $\quad [(a,b)]_\sim := \left\{ (x,y) \;\middle|\; (x,y) \sim (a,b) \right\}$

and $\quad (x,y) \sim (a,b) \quad :\Leftrightarrow \quad x + b = a + y$

$[(0,0)]_\sim =: 0_\mathbb{Z}$ $\qquad\qquad$ $[(0,1)]_\sim =: (-1)_\mathbb{Z}$

$[(1,0)]_\sim =: 1_\mathbb{Z}$ $\qquad\qquad$ $[(0,2)]_\sim =: (-2)_\mathbb{Z}$

$[(2,0)]_\sim =: 2_\mathbb{Z}$ $\qquad\qquad$ $\vdots$

$\vdots$

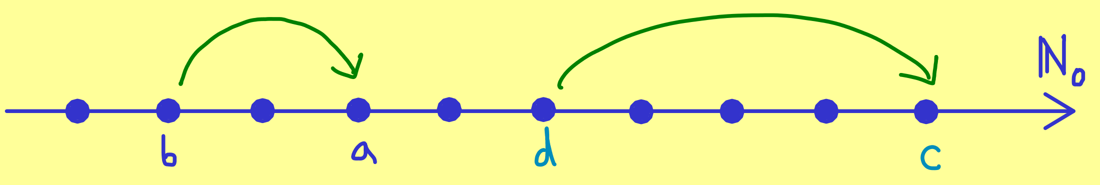
$$\mathbb{Z} = \left\{ \dots, (-2)_\mathbb{Z}, (-1)_\mathbb{Z}, 0_\mathbb{Z}, 1_\mathbb{Z}, 2_\mathbb{Z}, \dots \right\}$$

Question: $\quad$ Is $\quad 4_\mathbb{Z} + x = 0_\mathbb{Z} \quad$ now solvable? And with $\; x = (-4)_\mathbb{Z}$?

First question: $\quad$ How is $+$ as a map $\; \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \;$ defined?

$$[(a,b)]_\sim + [(c,d)]_\sim := [(a+c, b+d)]_\sim$$

well-defined? ✓



Take $\; (\tilde{a},\tilde{b}) \sim (a,b) \;$ and $\; (\tilde{c},\tilde{d}) \sim (c,d)$. Then $\; [(\tilde{a},\tilde{b})]_\sim + [(\tilde{c},\tilde{d})]_\sim = [(\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d})]_\sim$

Is $\; (\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d}) \sim (a+c, b+d)$?

Proof: $\quad (\tilde{a},\tilde{b}) \sim (a,b) \;\Leftrightarrow\; \tilde{a} + b = a + \tilde{b}$

$\qquad\quad (\tilde{c},\tilde{d}) \sim (c,d) \;\Leftrightarrow\; \tilde{c} + d = c + \tilde{d}$

implies: $\; \tilde{a} + \tilde{c} + b + d = a + c + \tilde{b} + \tilde{d}$

$\Leftrightarrow \; (\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d}) \sim (a+c, b+d)$

Examples: (a) $\quad 4_\mathbb{Z} + 2_\mathbb{Z} = [(4,0)]_\sim + [(2,0)]_\sim = [(6,0)]_\sim = 6_\mathbb{Z}$

(b) $\quad 4_\mathbb{Z} + (-4)_\mathbb{Z} = [(4,0)]_\sim + [(0,4)]_\sim = [(4,4)]_\sim = [(0,0)]_\sim = 0_\mathbb{Z}$

Properties of $\mathbb{Z}$ together with $+$: $\quad$ map $\; \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

(a) associative

(b) commutative

(c) $m + 0_\mathbb{Z} = m \qquad$ ($0_\mathbb{Z}$ is neutral element)

(d) For all $m \in \mathbb{Z}$, there is an element $\tilde{m} \in \mathbb{Z}$ with $\; m + \tilde{m} = 0_\mathbb{Z}$

$(\mathbb{Z}, +)$ is an abelian group