The Bright Side of Mathematics

Start Learning Numbers – Part 5

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

$$\underbrace{4+4+4+4+4}_{\text{We have } 5 \text{ of them}} =: 5 \cdot 4$$

$$3+3+3+3+3+3 =: 6 \cdot 3$$

$$4 =: 1 \cdot 4$$

$$0 =: 0 \cdot 4$$

How can we define the multiplication?

Multiplication in $\mathbb{N}_0$: map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$

$(n, m) \longmapsto n \cdot m$ defined by

$$0 \cdot m := 0$$
$$(n+1) \cdot m := (n \cdot m) + m$$

(recursive definition)

$$5 \cdot 2 = 2 + 2 + 2 + 2 + 2$$
$$6 \cdot 2 = \underbrace{2+2+2+2+2}_{5 \cdot 2} + 2$$

successor

(Map is well-defined by Dedekind's recursion theorem)

Properties:

(1) $n \cdot (m \cdot k) = (n \cdot m) \cdot k$ (associative)

(2) $n \cdot m = m \cdot n$ (commutative)

(3) $1 \cdot m = m$ (neutral element)

How to connect + and $\cdot$: $n \cdot (m+k) = n \cdot m + n \cdot k$ (distributive)

$$\boxed{\begin{aligned} 0 \cdot m &:= 0 \\ (n+1) \cdot m &:= (n \cdot m) + m \end{aligned}} \quad (*)$$

Proof by induction: Base case: $n = 0$

Left-hand side: $0 \cdot (m+k) = 0$ ✓

Right-hand side: $0 \cdot m + 0 \cdot k = 0 + 0 = 0$ ✓

Induction step: Assume $n \cdot (m+k) = n \cdot m + n \cdot k$ holds for n.

(induction hypothesis)

Left-hand side: $(n+1) \cdot (m+k) \overset{(*)}{=} n \cdot (m+k) + (m+k)$

$\overset{(i.h.)}{=} n \cdot m + (n \cdot k + m) + k$

$= (n \cdot m + m) + (n \cdot k + k)$

$\overset{(*)}{=} (n+1) \cdot m + (n+1) \cdot k$ ← Right-hand side