The Bright Side of

Mathematics

Start Learning Numbers – Part 4

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Addition $+$ is a map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ with:

- $m + 0 = m$ (neutral element)
- $(k + m) + n = k + (m + n)$ (associative law)
- $m + n = n + m$ (commutative law)

Ordering: We write $n \leq m$ if:

$0 < 1 < 2 < 3 < 4 < 5$

$\exists k \in \mathbb{N}_0 : \quad m = n + k$

And we write $n < m$ if: $\quad n \leq m \quad \wedge \quad n \neq m$

Properties:

(1) $n \leq n$ (reflexive)

(2) If $n \leq m \ \wedge \ m \leq n$, then $n = m$ (antisymmetric)

(3) If $n \leq \ell \ \wedge \ \ell \leq m$, then $n \leq m$ (transitive)

Proof: Assume $n \leq \ell$ and $\ell \leq m$ are true. So:

$\exists k_1 \in \mathbb{N}_0 : \ \ell = n + k_1$ and $\exists k_2 \in \mathbb{N}_0 : \ m = \ell + k_2$ are true.

Therefore:
$$m = \ell + k_2 = (n + k_1) + k_2 = n + \underbrace{(k_1 + k_2)}_{=: k \in \mathbb{N}_0} = n + k$$

Therefore: $\exists k \in \mathbb{N}_0 : \ m = n + k$ is true, so $n \leq m$ is true.