The Bright Side of Mathematics

Start Learning Numbers – Part 3

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Each $n \in \mathbb{N}_0$ has a unique successor:

$s : \mathbb{N}_0 \to \mathbb{N}_0 \quad , \qquad s(n) = n + 1$

We already know: $m + (n+1) = (m+n) + 1$ (RD)

Mathematical induction:

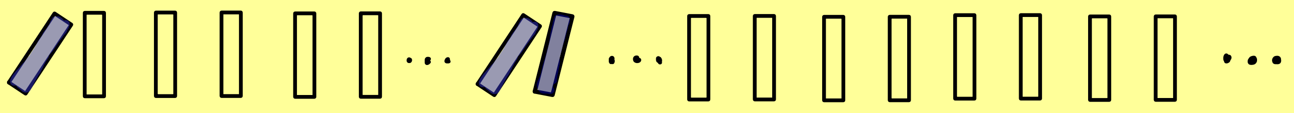
$\mathbb{N}_0$ satisfies the induction property:

Let $P(n)$ be a property for natural numbers n ("predicate").

If: (1) $P(0)$ is true (base case)

(2) $\forall n \in \mathbb{N}_0 : P(n) \to P(n+1)$ is true (induction step)

Then: $P(n)$ is true for all $n \in \mathbb{N}_0$ $\quad (\forall n : P(n)$ is true $)$



0   n $n+1$

(base case)   (induction step)

Proposition: For all $k, m, n \in \mathbb{N}_0$, we have:

$$(k + m) + n = k + (m + n) \qquad \text{(associative law)}$$

Proof: Use mathematical induction.

$P(n)$ is given by:

$$\forall k, m \in \mathbb{N}_0 : \quad (k+m) + n = k + (m+n)$$

Base case: $P(0)$ means $\forall k, m \in \mathbb{N}_0 : \quad \underbrace{(k+m)+0}_{k+m} = k + (\underbrace{m+0}_{m})$

$\Leftrightarrow \quad \forall k, m \in \mathbb{N}_0 : \quad k + m = k + m$ true

Induction step: $(\forall n \in \mathbb{N}_0 : P(n) \to P(n+1))$

$\boxed{m + (n+1) = (m+n) + 1 \quad \text{(RD)}}$

Assume $P(n)$ is true.

$P(n+1)$ means $\forall k, m \in \mathbb{N}_0 : \quad (k+m) + (n+1) = k + (m + (n+1))$

Left-hand side: $(k+m) + (n+1) \overset{\text{(RD)}}{=} ((k+m)+n) + 1$

$\overset{P(n)}{=} (k + (m+n)) + 1$

Right-hand side

$\overset{\text{(RD)}}{=} k + ((m+n)+1) \overset{\text{(RD)}}{=} k + (m + (n+1))$