The Bright Side of Mathematics

Real Analysis – Part 20

$\sum_{k=1}^{\infty} a_k$ absolutely convergent?

There is a convergent majorant $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

We know the geometric series!

$$\sum_{k=0}^{\infty} q^k \text{ convergent} \iff |q| < 1$$

Fact: If there is $n_0 \in \mathbb{N}$ and $C, q \in \mathbb{R}$ with $|q| < 1$ such that $|a_k| \leq C \cdot q^k$ for all $k \geq n_0$,

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Ratio test: If there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$$a_k \neq 0 \quad \text{and} \quad \left|\frac{a_{k+1}}{a_k}\right| \leq q \quad \text{for all } k \geq n_0,$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Proof: $|a_{k+1}| \leq q \cdot |a_k| \leq q \cdot q \cdot |a_{k-1}| \leq \overset{\text{inductively}}{\cdots} \leq q^{k+1-n_0} |a_{n_0}| = q^{k+1} \underbrace{\frac{|a_{n_0}|}{q^{n_0}}}_{C}$

Example: $\sum_{k=1}^{\infty} \frac{1}{k!}$ convergent? $\left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{1}{k+1} \leq \underbrace{\frac{1}{2}}_{q}$ for all $k \geq \underbrace{1}_{n_0}$.

$\Rrightarrow$ Yes! (by ratio test) $\boxed{(k+1)! = (k+1) \cdot k!}$

Warning: $\left|\frac{a_{k+1}}{a_k}\right| < 1$ is not enough!

Root test: If there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$$\sqrt[k]{|a_k|} \leq q \quad \text{for all } k \geq n_0,$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Proof: $\sqrt[k]{|a_k|} \leq q \iff |a_k| \leq q^k$

Example: $\sum_{k=1}^{\infty} \left(\frac{3}{\sqrt{2+k}}\right)^{2k}$ convergent? $\sqrt[k]{\left(\frac{3}{\sqrt{2+k}}\right)^{2k}} = \left(\frac{3}{\sqrt{2+k}}\right)^2 = \frac{9}{2+k} \leq \frac{9}{10}$ for all $k \geq 8$

Yes, by root test!

Remember:

$$\limsup_{k \to \infty} \sqrt[k]{|a_k|} < 1 \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is abs. convergent!}$$

$$\limsup_{k \to \infty} \sqrt[k]{|a_k|} > 1 \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent!}$$