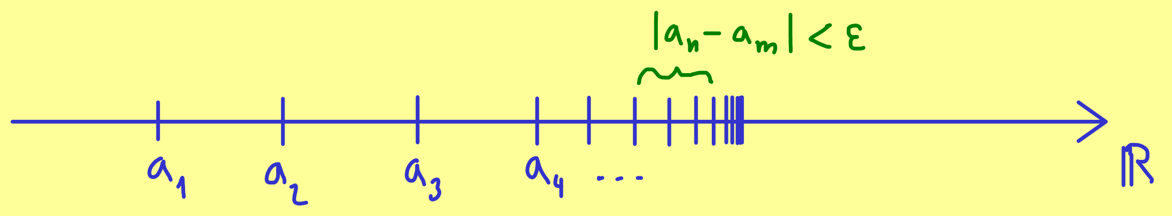
The Bright Side of Mathematics

Real Analysis – Part 7

$(a_n)_{n\in\mathbb{N}}$ convergent (there is a limit $a = \lim_{n\to\infty} a_n$)

Different idea:



Definition: If $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n, m \geq N : \; |a_n - a_m| < \varepsilon$,

then $(a_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence.

Important fact: For a sequence of real numbers:

Completeness axiom (C)

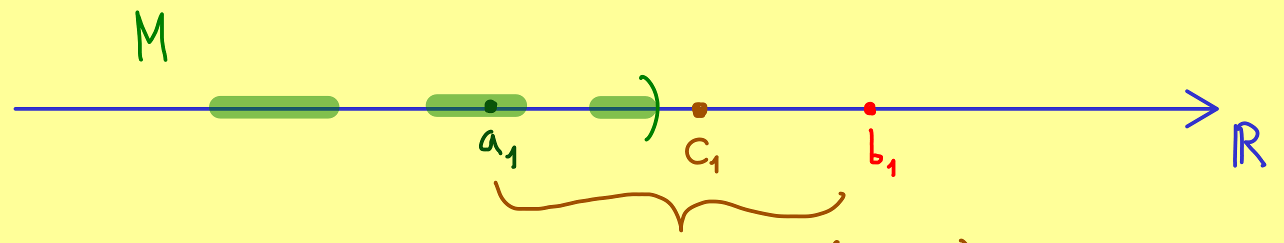
Cauchy sequence $\Longleftrightarrow$ Convergent sequence

Start Learning Real – Part 2

Dedekind completeness: If $M \subseteq \mathbb{R}$ is bounded from above, then $\sup M \in \mathbb{R}$ (exists)

If $M \subseteq \mathbb{R}$ is bounded from below, then $\inf M \in \mathbb{R}$ (exists)

Proof:



$c_1 := \frac{1}{2}(a_1 + b_1)$

Two cases: (1) c_1 is an upper bound for M : $b_2 := c_1$, $a_2 := a_1$

(2) c_1 is not an upper bound for M : $\exists x \in M : x > c_1$

$a_2 := x$, $b_2 := b_1$

$c_n := \frac{1}{2}(a_n + b_n)$

Two cases: (1) c_n is an upper bound for M : $b_{n+1} := c_n$, $a_{n+1} := a_n$

(2) c_n is not an upper bound for M : $\exists x \in M : x > c_n$

$a_{n+1} := x$, $b_{n+1} := b_n$

For $m > n$: $|b_n - b_m| \leq |b_n - a_n| \leq \underbrace{\left(\frac{1}{2}\right)^{n-1} |b_1 - a_1|}_{\text{gets arbitrarily small}}$

$\Longrightarrow$ $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence

$\Longrightarrow$ $(b_n)_{n\in\mathbb{N}}$ is a convergent sequence with limit $\sup M$

Important application: If $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing ($a_{n+1} \leq a_n$ for all n)

and bounded from below (the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound),

then: $(a_n)_{n\in\mathbb{N}}$ is convergent.