



The Bright Side of Mathematics

Functional analysis - part 27

Bounded inverse theorem: X, Y Banach spaces, $T \in \mathcal{B}(X, Y)$.

Then: T bijective $\Rightarrow T^{-1} \in \mathcal{B}(Y, X)$ (It's continuous)

Proof: T bijective $\Rightarrow T$ open map $\Rightarrow T^{-1}$ continuous \square
open mapping theorem

Counterexample: $X = (C([0,1]), \|\cdot\|_\infty)$, $Y = (\{f \in C^1([0,1]) \mid f(0)=0\}, \|\cdot\|_\infty)$ \rightarrow not complete

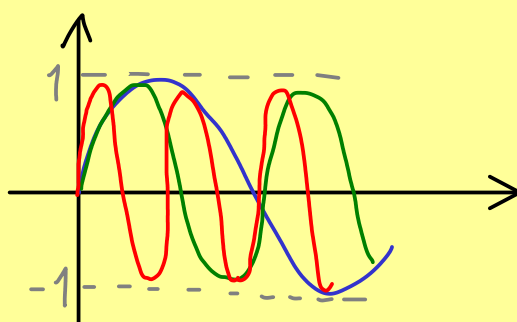
$$(Tf)(t) = \int_0^t f(s) ds \quad \text{linear and bounded and bijective}$$

$$\|Tf\|_\infty = \sup_{t \in [0,1]} \left| \int_0^t f(s) ds \right| \leq \|f\|_\infty \quad \Rightarrow \quad \|T\|_{X \rightarrow Y} \leq 1$$

Take $f_k(t) = \sin(kt)$

$$(Tf_k)(t) = \frac{1}{k} (1 - \cos(kt))$$

$\underbrace{\hspace{10em}}_{g_k(t)}$



$$T^{-1}g_k = f_k$$

$$\Rightarrow \|T^{-1}\|_{Y \rightarrow X} \geq \frac{\|T^{-1}g_k\|_\infty}{\|g_k\|_\infty} = \frac{\|f_k\|_\infty}{\|g_k\|_\infty} \geq \frac{k}{2} \xrightarrow{k \rightarrow \infty} \infty$$

$\Rightarrow T^{-1}$ not continuous