



# The Bright Side of Mathematics

## Functional analysis - part 23

Dual space:  $X$  normed space

$$X' := \left\{ \ell: X \rightarrow \mathbb{F} \mid \ell \text{ linear + bounded} \right\}$$

Example:  $X = \ell^p(\mathbb{N})$  for  $p \in (1, \infty)$

$X' \cong \ell^{p'}(\mathbb{N})$  where  $p' \in (1, \infty)$  Hölder conjugate  $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$   
 there is an isometric isomorphism

$$T: \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'$$

$$(Tx)(y) := \sum_{k=1}^{\infty} x_k \cdot y_k \quad \text{or} \quad x \mapsto \langle \bar{x}, \cdot \rangle_{\ell^p(\mathbb{N})}$$

To show: (1)  $T$  is well-defined ✓ (4)  $T$  surjective  
 (2)  $T$  is linear ✓ (5)  $\|Tx\| = \|x\|$  for all  $x \in \ell^{p'}(\mathbb{N})$   
 (3)  $T$  is bounded ✓ (isometric)

Proof: (1)  $|(Tx)(y)| \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k \cdot x_k| \stackrel{\text{Hölder}}{\leq} \|y\|_p \cdot \|x\|_{p'} < \infty$

$\Rightarrow Tx$  is linear and bounded for all  $x \in \ell^{p'}(\mathbb{N})$

(2)  $T$  is linear.

$$(3) \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} = \sup \left\{ |(Tx)(y)| \mid \|y\|_p = 1 \right\} \leq \|x\|_{p'} \leq \|y\|_p \cdot \|x\|_{p'}$$

$$T: \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'$$

$$\Rightarrow \|T\| \leq 1$$

(4) Let  $y' \in (\ell^p(\mathbb{N}))'$  and  $e_k = (0, 0, \dots, 0, 1, 0, \dots)$ . k<sup>th</sup> position

Define:  $x_k := y'(e_k)$  and  $x := (x_k)_{k \in \mathbb{N}}$

Question:  $x \in \ell^{p'}(\mathbb{N})$  and  $Tx = y'$ ?

$$\sum_{k=1}^n |x_k|^{p'} = \sum_{k=1}^n x_k \cdot t_k \quad \left\{ \begin{array}{l} \frac{|x_k|^{p'}}{x_k}, \quad x_k \neq 0 \\ 0, \quad x_k = 0 \end{array} \right.$$

$$= \sum_{k=1}^n t_k \cdot y'(e_k) = y' \left( \sum_{k=1}^n t_k e_k \right)$$

$$\leq \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot \left\| \sum_{k=1}^n t_k e_k \right\|_p = \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot \left( \sum_{k=1}^n |t_k|^p \right)^{\frac{1}{p}}$$

$$= \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot \left( \sum_{k=1}^n |x_k|^{p'} \right)^{\frac{1}{p}} = \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot |x_k|^{(p'-1) \cdot \frac{1}{p}} = \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot |x_k|^{p' - 1}$$

$$\left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$$

$$\stackrel{n \rightarrow \infty}{\Rightarrow} \|x\|_{p'} \leq \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \Rightarrow x \in \ell^{p'}(\mathbb{N}) \quad \checkmark$$

$$\text{For } y' \in (\ell^p(\mathbb{N}))': (Tx - y')(y) = (Tx - y') \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n y_k e_k \right)$$

$$(Tx)(y) := \sum_{j=1}^{\infty} x_j \cdot y_j$$

$$\stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} (Tx - y') \left( \sum_{k=1}^n y_k e_k \right)$$

$$\stackrel{\text{linearity}}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n y_k (Tx - y')(e_k) = 0 \quad \text{surjective} \checkmark$$

$$(5) \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \leq \|x\|_{p'} \leq \|y'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} = \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \quad \text{isometry} \checkmark$$