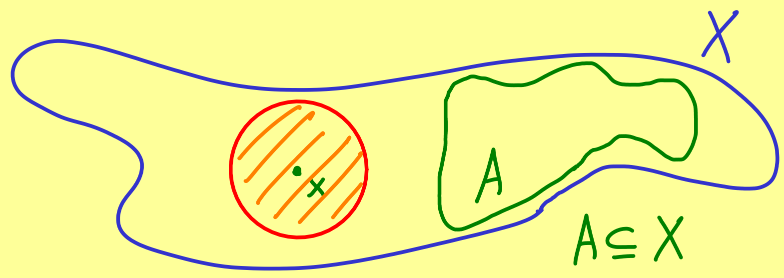
The Bright Side of Mathematics

Functional analysis – part 3

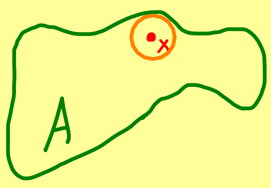
(X, d) metric space



$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$ (open ball of radius $\varepsilon > 0$ centered at x)

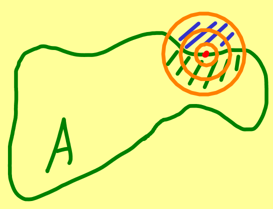
Notions:

(1) Open sets:



$A \subseteq X$ is called <u>open</u> if for each $x \in A$ there is an open ball with $B_\varepsilon(x) \subseteq A$.

(2) Boundary points:

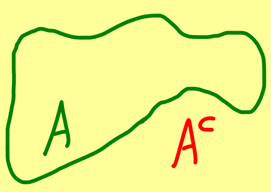


$A \subseteq X$. $x \in X$ is called a <u>boundary point for A</u> if for all $\varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$ $[A^c := X \setminus A]$

$\partial A := \{ x \in X \mid x \text{ is boundary point for } A \}$

Remember: A open $\Longleftrightarrow$ $A \cap \partial A = \emptyset$

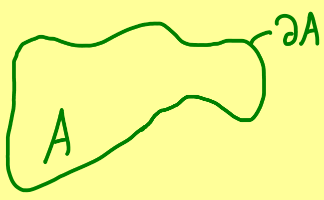
(3) Closed sets:



$A \subseteq X$ is called <u>closed</u> if $A^c := X \setminus A$ is open.

Remember: A closed $\Longleftrightarrow$ $A \cup \partial A = A$

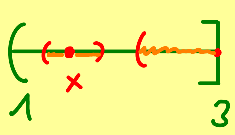
(4) Closure:



$\overline{A} := A \cup \partial A$ (always closed!)

Example: $X := (1, 3] \cup (4, \infty)$, $d(x,y) := |x-y|$, (X, d) is a metric space

(a) $A := (1,3] \subseteq X$ open?



For $x \in A$, $x \neq 3$, define $\varepsilon := \frac{1}{2}\min(|1-x|, |3-x|)$. Then $B_\varepsilon(x) \subseteq A$.
For $x = 3$: $B_1(x) = \{ y \in X \mid d(x,y) < 1 \} = (2, 3] \subseteq A$
$\Rightarrow$ A is open

(b) A is also closed!

(c) $C := (1,2]$, $\partial C = \{2\}$, $\overline{C} = C$