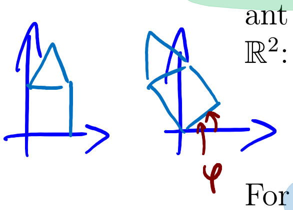
$$
\begin{aligned}
&\overset{(8.20)}{\iff} && (\ell - \lambda\, id)_{\mathcal{B}\leftarrow\mathcal{B}} = \ell_{\mathcal{B}\leftarrow\mathcal{B}} - \lambda\, id_{\mathcal{B}\leftarrow\mathcal{B}} = \ell_{\mathcal{B}\leftarrow\mathcal{B}} - \lambda \mathbb{1} \\
& && \text{is not invertible for any basis } \mathcal{B} \text{ of } V \\
&\iff && \lambda \text{ is an eigenvalue of } \ell_{\mathcal{B}\leftarrow\mathcal{B}} \text{ for all bases } \mathcal{B} \text{ of } V \\
&\iff && \det\big((\ell - \lambda\, id)_{\mathcal{B}\leftarrow\mathcal{B}}\big) = 0 \text{ for all bases } \mathcal{B} \text{ of } V \\
&\iff && \det(\ell - \lambda\, id) = 0
\end{aligned}
$$

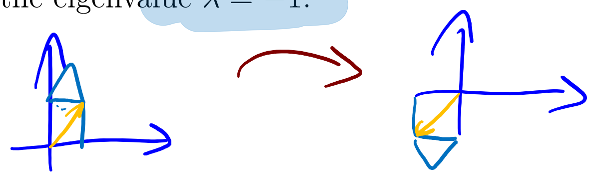
VL23 ↓

Example 8.31. (a) The rotation $d \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ from Example 8.3 (e) has the determinant 1 since the associated matrix representation(8.12) w.r.t. the standard basis $\mathcal{B}$ in $\mathbb{R}^2$:



$$\det(d) = \det(d_{\mathcal{B}\leftarrow\mathcal{B}}) = \det\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = (\cos\varphi)^2 + (\sin\varphi)^2 = 1.$$

For $\mathbb{F} = \mathbb{R}$, we only find eigenvalues and eigenvector if φ is an integer multiple of π. For example, for $\varphi = \pi$, we have $d = -id$ and hence each vector in $\mathbb{R}^2$ is an eigenvector for the eigenvalue $\lambda = -1$.



(b) For the orthogonal projection $\mathrm{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ onto the line $G := \mathrm{Span}(\mathbf{n})$ and both variants

$$\mathrm{proj}_E = id - \mathrm{proj}_G \qquad \text{and} \qquad \mathrm{refl}_E = id - 2\,\mathrm{proj}_G$$

from Example 8.7, 8.9, 8.14 (f) and 8.17 (a), we find with the help of equation (8.13):

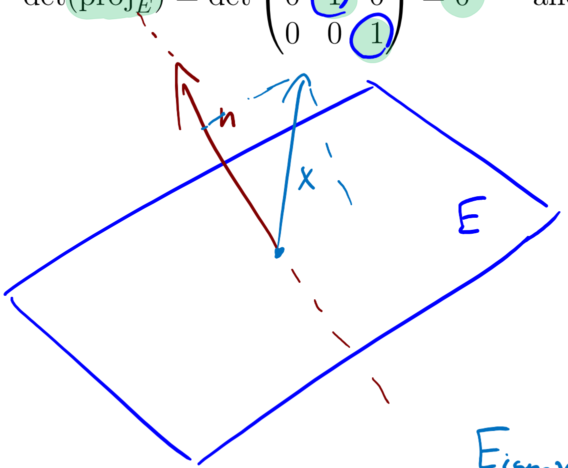
$$\det(\mathrm{proj}_G) = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

(proj_G is not invertible)

Using Example 8.17 (a), we get:

$$\det(\mathrm{proj}_E) = \det\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \qquad \text{and} \qquad \det(\mathrm{refl}_E) = \det\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1.$$

Eigenvalues



Eigenvectors for eigenvalue 1:

For proj_G = $G \setminus \{0\}$

For proj_E = $E \setminus \{0\}$

Eigenvectors for eigenvalue 0:

For proj_G = $E \setminus \{0\}$,

For proj_G each vector from G is an eigenvector for the eigenvalue 1, and each vector

$\mathrm{spec}(\mathrm{refl}_E) = \{-1, 1\}$

from E is an eigenvector for the eigenvalue 0 since E is the kernel of proj_G. For proj_E we have the same with $G \leftrightarrow E$. For refl_E each vector from G is an eigenvector for the eigenvalue -1, and each vector from E is an eigenvector for the eigenvalue 1.

Summary

- A map ℓ from one $\mathbb{F}$-vector space V to another $\mathbb{F}$-vector space W is called *linear* if $\ell(\mathbf{x}+\mathbf{y}) = \ell(\mathbf{x}) + \ell(\mathbf{y})$ and $\ell(\alpha\mathbf{x}) = \alpha\ell(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$. We write: $\ell \in \mathcal{L}(V, W)$.
- Linear maps $\mathcal{L}(V, W)$ can be added and scaled with $\alpha \in \mathbb{F}$. Hence, $\mathcal{L}(V, W)$ gets an $\mathbb{F}$-vector space.
- The composition $k \circ \ell$ of linear maps $\ell : U \to V$ and $k : V \to W$ is linear.
- The inverse map of a bijective linear map is again linear. Therefore a bijective linear map is called an *isomorphism*.
- Each linear map $\ell \in \mathcal{L}(V, W)$, between finite dimensional vector spaces V and W, can be identified with a matrix. In order to do this, choose a basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ in V and a basis $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ in W. Be using the basis isomorphisms $\Phi_\mathcal{B}$ and $\Phi_\mathcal{C}$, we get a linear map $\mathbb{F}^n \to \mathbb{F}^m$. Such a linear map is also represented by a $m \times n$ matrix $\ell_{\mathcal{C}\leftarrow\mathcal{B}} := (\ell(\mathbf{b}_1)^\mathcal{C} \cdots \ell(\mathbf{b}_n)^\mathcal{C})$. It is called the *matrix representation* of ℓ w.r.t. $\mathcal{B}$ and $\mathcal{C}$.
- The matrix representation of $k + \ell$ is the sum of both matrix representations.
- The matrix representation of $\alpha\ell$ is α times the matrix representation of ℓ.
- The matrix representation of $k \circ \ell$ is the product of both matrix representations.
- The matrix representation of ℓ^{-1} is the inverse of the matrix representation of ℓ.
- Kernel and range of a linear map ℓ can be calculated by $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$.
- By changing the basis of V from $\mathcal{B}$ to $\mathcal{B}'$ and changing the basis of W from $\mathcal{C}$ to $\mathcal{C}'$, the matrix representation of $\ell : V \to W$ changes from $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$ to $\ell_{\mathcal{C}'\leftarrow\mathcal{B}'}$. In this case, we have $\ell_{\mathcal{C}'\leftarrow\mathcal{B}'} = T_{\mathcal{C}'\leftarrow\mathcal{C}}\, \ell_{\mathcal{C}\leftarrow\mathcal{B}}\, T_{\mathcal{B}\leftarrow\mathcal{B}'}$.
- We call two matrices A and B *equivalent* and write $A \sim B$ if there are invertible matrices S and T with $B = SAT$.
- We have $A \sim B$ if and only if $\operatorname{rank}(A) = \operatorname{rank}(B)$.
- For the special case $\ell : V \to V$, one often chooses the same basis $\mathcal{B}$ left and right. How does the matrix $\ell_{\mathcal{B}\leftarrow\mathcal{B}}$ change when changing the basis $\mathcal{B}$ to $\mathcal{B}'$? Then, we have $S = T^{-1}$ in the formula above.
- Two matrices A and B are called *similar* and one writes $A \approx B$ if there is an invertible matrix T with $B = T^{-1}AT$.
- From $A \approx B$ follows $\det(A) = \det(B)$ and $\operatorname{spec}(A) = \operatorname{spec}(B)$ but the converse is in general false.
- $\det(\ell)$ for a linear map $\ell : V \to V$ is defined by $\det(\ell_{\mathcal{B}\leftarrow\mathcal{B}})$ for any basis $\mathcal{B}$ in V.
- $\lambda \in \mathbb{F}$ is an *eigenvalue* of $\ell : V \to V$ if $\ell(\mathbf{x}) = \lambda\mathbf{x}$ for some $\mathbf{x} \in V \setminus \{\mathbf{o}\}$.

9 Some matrix decompositions

$\ell: V \to W$ ⇝ matrix representations $\ell_{C \leftarrow B}$ ⇝ Examine matrices

decompositions makes your life easier

(change of basis)

- Chapter 3: $A \in \mathbb{R}^{n\times n}$ ⇝ $A = LU$

  $A \in \mathbb{R}^{m\times n}$ ⇝ $A = PLK$ $=$ ( ) $\cdot$ ( )

  row echelon form

  easier

  works the same for $\mathbb{C}^{m\times n}$

- Chapter 5: $A \in \mathbb{R}^{n\times n}$ ⇝ $A = Q \cdot R$ $=$ (| | |) $\cdot$ ( )

  columns ONB

  (orthogonal $Q^T = Q^{-1}$)

  works the same for $A \in \mathbb{C}^{m\times n}$, $m \geq n$

  rank(A)

  $A = Q \cdot R = \underbrace{(\,|\;|\;|\,)}_{\text{ONS}} \cdot \underbrace{(\;\;)}_{n}\Big\}n$

  ONS (w.r.t. standard inner product)

- Chapter 6: Diagonalisation: $A \in \mathbb{C}^{n\times n}$

  $A = XDX^{-1}$, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

  eigenvectors

9.1 Jordan normal form

We are searching for the best substitute of the usual diagonalisation $A = XDX^{-1}$ such that it works for all matrices $A \in \mathbb{C}^{n\times n}$. A good thing would be to use a triangular matrix instead of D if A is not diagonalisable. The next Proposition tells us that we only need some 1s above the diagonal:

diagonalisable ⟺ similar to diagonal matrix $(D = X^{-1}AX)$

Now: Each $A \in \mathbb{C}^{n\times n}$ is similar to a triangular matrix

Proposition & Definition 9.1. Jordan normal form

Let $A \in \mathbb{C}^{n\times n}$ with pairwise different eigenvalues $\lambda_1, \dots, \lambda_r \in \mathbb{C}$, where $\alpha_1, \dots, \alpha_r$ denote the corresponding algebraic multiplicities and $\gamma_1, \dots, \gamma_r$ the corresponding geometric multiplicities. Then, there is an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that

$$A = XJX^{-1} \quad \text{or equivalently} \quad X^{-1}AX = J$$

some 1s ?

$$J = \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & \ddots & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \boxed{\begin{matrix}\lambda_r & & \\ & \lambda_r & \\ & & \lambda_r\end{matrix}} \end{pmatrix}$$

and $J \in \mathbb{C}^{n\times n}$ has the following block diagonal form:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix}.$$

J is called a Jordan normal form (JNF) of A. The entries J_i are again block matrices, which are called Jordan blocks, and have the following structure:

$$J_i = \begin{pmatrix} J_{i,1} & & \\ & \ddots & \\ & & J_{i,\gamma_i} \end{pmatrix} \in \mathbb{C}^{\alpha_i \times \alpha_i},$$

For each eigenvalue λ_i one block with size α_i

where the matrices $J_{i,\ell}$ are called Jordan boxes and have the following form:

$$J_{i,\ell} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

$J_{i,\ell} = (\lambda_i)$

Note that $J_{i,\ell}$ could also be a 1×1-matrix.

For $\alpha_i = \gamma_i$, $J_i = \underbrace{\begin{pmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{pmatrix}}_{\text{diagonal}}$

Proof: Literature! (Not easy!)

$$\begin{pmatrix} \lambda_1 & 1 \\ & \lambda_1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & \\ & \lambda_1 \end{pmatrix}$$

Example 9.2. If you have a matrix $A \in \mathbb{C}^{9\times 9}$ and find an invertible matrix X with $A = XJX^{-1}$ such that

$$J = \begin{pmatrix} \underbrace{\begin{array}{ccc|cc} 4 & 1 & & & \\ & 4 & 1 & & \\ & & 4 & & \\ \hline & & & 4 & 1 \\ & & & & 4 \end{array}}_{J_1} & \\ & \overbrace{\begin{array}{cc|c|c} -3 & 1 & & \\ & -3 & & \\ \hline & & -3 & \\ \hline & & & -3 \end{array}}^{J_2} \end{pmatrix},$$

$\lambda_1 = 4$

$\lambda_2 = -3$

then you immediately find the following informations for A:

- $\mathrm{spec}(A) = \{4, -3\}$, $\alpha_1 = 5$, $\alpha_2 = 4$
- Two boxes for λ_1: $\gamma_1 = 2$
- Three boxes for λ_2: $\gamma_2 = 3$

$\rightsquigarrow$ We need more information than α_i, γ_i

On the other hand, we learn that J is not determined solely by eigenvalues and multiplicities because also the matrix

$$J' = \begin{pmatrix} \begin{array}{cccc|c} 4 & 1 & & & \\ & 4 & 1 & & \\ & & 4 & 1 & \\ & & & 4 & \\ \hline & & & & 4 \end{array} & \\ & \begin{array}{cc|c|c} -3 & 1 & & \\ & -3 & & \\ \hline & & -3 & \\ \hline & & & -3 \end{array} \end{pmatrix}$$

would fit to these parameters above

$$\lambda_1 = 4,\ \alpha_1 = 5,\ \gamma_1 = 2, \qquad \lambda_2 = -3,\ \alpha_2 = 4,\ \gamma_2 = 3.$$

there is no invertible C with: $(CJ' = JC)$

J' not similar to J!

Example:

$$\begin{pmatrix} \begin{array}{ccc|c} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ \hline & & & 0 \end{array} \end{pmatrix} \not\sim \begin{pmatrix} \begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 1 \\ & & 0 & 0 \end{array} \end{pmatrix}$$

since

zero column

$$\begin{pmatrix} 0 & C_{11} & C_{12} & 0 \\ 0 & C_{21} & C_{22} & 0 \\ 0 & C_{31} & C_{32} & 0 \\ 0 & C_{41} & C_{42} & 0 \end{pmatrix} = \begin{pmatrix} C_{21} & C_{22} & C_{23} & C_{24} \\ 0 & 0 & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

→ Part of the proof

Construction of J and X: How and why?

First we need the eigenvalues $\lambda_1, \ldots, \lambda_r$ of A because on a triangular matrix they have to be on the diagonal counted with the algebraic multiplicities. So we also determine $\alpha_1, \ldots, \alpha_r$. For each λ_i, we do the following procedure.

Rule of thumb: Treat the problem for all λ_i separately.

Each λ_i has its own Jordan block J_i and corresponding columns in X. Therefore, we can deal with the problem for each eigenvalue separately and put it together in the end.

Since $A \approx J$, we already know that the characteristic polynomial of A and J coincide (cf. Proposition 6.26). Hence, both matrices have the same eigenvalues with the same algebraic multiplicities. They have to be on the diagonal of J by Proposition 6.9.

Size of J_i

The Jordan block J_i for the eigenvalue λ_i has the size $\alpha_i \times \alpha_i$ because we need λ_i as often on the diagonal of J as the algebraic multiplicity says.

The n columns of X have to be linearly independent vectors from $\mathbb{C}^n$ in order that X is invertible. Just looking at the $\alpha_i \times \alpha_i$-block J_i, we need α_i columns from this matrix X. How to get them?

Recall that for the diagonalisation, in the case that A is diagonalisable, we had enough eigenvectors corresponding to the eigenvalue λ_i, which means vectors from $\operatorname{Ker}(A - \lambda_i \mathbb{1})$. We could choose them as a linearly independent family because

$$\dim(\operatorname{Ker}(A - \lambda_i \mathbb{1})) =: \gamma_i = \alpha_i.$$ ← diagonalisable

In the case $\gamma_i < \alpha_i$ (which means A is not diagonalisable), we are missing some columns in X.

To shorten everything: $A - \lambda_i \mathbb{1} =: N$

Let us look at an example with $\alpha_i = 8$ and $\gamma_i = 4$. Choose $\mathbf{x}_1, \ldots, \mathbf{x}_4 \in \operatorname{Ker}(N)$, which are eigenvectors of A.

$\operatorname{Ker}(N)$
dimension: $4 = \gamma_i$ $\quad \mathbf{x}_{1,1} \quad \mathbf{x}_{2,1} \quad \mathbf{x}_{3,1} \quad \mathbf{x}_{4,1}$

We need $\alpha_i = 8$ linearly independent vectors for X but at this point we only have $\gamma_i = 4$. How to get the missing four vectors?

Answer: Since we have not found enough vectors in the kernel of N, we can look at the kernels of $N^2, N^3, \ldots$ until we have found 8 vectors in total. Clearly:

$$\operatorname{Ker}(N) \subset \operatorname{Ker}(N^2) \subset \operatorname{Ker}(N^3) \subset \cdots, \text{ since } N\mathbf{x} = \mathbf{o} \Rightarrow N^2\mathbf{x} = N(N\mathbf{x}) = \mathbf{o}, \ldots$$

Recall: $\operatorname{Ker}(N)$ has the dimension $\gamma_i = 4$. Suppose that $\operatorname{Ker}(N^2)$ is of dimension 7 and that $\operatorname{Ker}(N^3)$ has dimension $8 = \alpha_i$. The difference

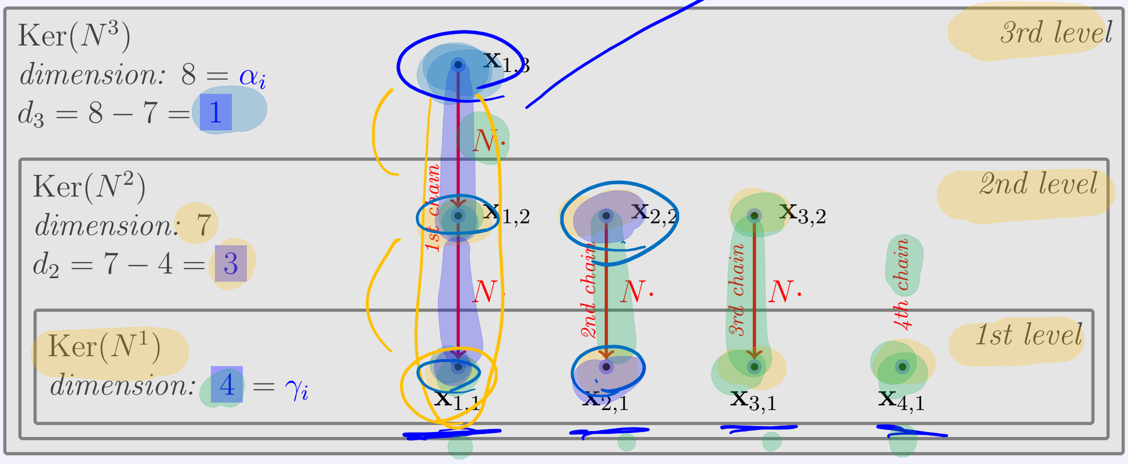
$$d_k := \dim(\operatorname{Ker}(N^k)) - \dim(\operatorname{Ker}(N^{k-1}))$$

$d_1 = 4 \ (\gamma_i)$ $\qquad d_2 = 3, \quad d_3 = 1$

eigenvectors $\qquad$ generalised eigenvectors of rank 2 or 3 of the dimensions shows us where to find the four missing vectors. Elements from the spaces $\mathrm{Ker}(N^k)$ are called *generalised eigenvectors*. To be more clear, we call an element from $\mathrm{Ker}(N^k) \setminus \mathrm{Ker}(N^{k-1})$ a *generalised eigenvector of rank k*. In this sense, the ordinary eigenvectors are now generalised eigenvector of rank 1.

chains correspond to boxes

Box 9.3. Levels and Jordan chains



As you can see in the picture, the vectors form “chains”, from top to bottom. We call each of these sequences a *Jordan chain* and it will be related to a Jordan box.

Box 9.4. Number and size of the Jordan boxes

Each Jordan chain ends at an ordinary eigenvector $\mathbf{x}_{j,1} \in \mathrm{Ker}(N)$. Therefore, we have exactly γ_i Jordan boxes inside the chosen Jordan block J_i. The length of a Jordan chain is the size of the corresponding Jordan box. All sizes add up to α_i (here: 8), which is exactly the size of the Jordan block J_i.

Looking at our example, we have 4 Jordan boxes of size 3, 2, 2 and 1. Hence:

$$J_i = \mathrm{Diag}\left(\begin{pmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 \\ & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 \\ & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i \end{pmatrix} \right) \in \mathbb{C}^{8\times 8}.$$

At this point, we now know the whole block J_i. The next step is to find the corresponding columns of X, which means that we have to calculate the generalised eigenvectors $\mathbf{x}_{j,k}$:

Box 9.5. Generalised eigenvectors: Start the Jordan chain

The starting point $\mathbf{x}_{j,k}$ for the jth Jordan chain can be chose in an almost arbitrary way from the kth level: Let $\mathbf{x}_{j,k} \in \mathrm{Ker}(N^k)$, but

$$\mathbf{x}_{j,k} \notin \mathrm{Span}\big(\mathrm{Ker}(N^{k-1}) \cup \{\mathbf{x}_{1,k}, \dots, \mathbf{x}_{j-1,k}\}\big), \tag{9.1}$$

where $\mathbf{x}_{1,k}, \dots, \mathbf{x}_{j-1,k}$ are the vectors from the chains before, 1 to $j-1$, which lie on the same level k. Now you can build the whole chain to the bottom $\mathbf{x}_{j,1}$. We just have to multiply with N in each step:

For $\mathbf{x} \in \mathrm{Ker}(N^k)$, we have $N\mathbf{x} \in \mathrm{Ker}(N^{k-1})$ since $\mathbf{o} = N^k\mathbf{x} = N^{k-1}(N\mathbf{x})$.

Note that equation (9.1) guarantees that all generalised eigenvectors on the kth level are start with eigenvector

linearly independent and that the linear independence remains on the levels below. All these α_i generalised eigenvectors are put as columns into X.

$X_i = \begin{pmatrix} | & | & \\ x_{11} & x_{12} & \cdots \\ | & | & \end{pmatrix}$

Box 9.6. Columns of X regarding λ_i

Let $X_i \in \mathbb{C}^{n \times \alpha_i}$ the matrix with columns filled out from left to right:

1st Jordan chain (bottom to top), ..., γ_ith Jordan chain (bottom to top).

For our example, this means: $X_i = \big(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}, \ \mathbf{x}_{2,1}, \mathbf{x}_{2,2}, \ \mathbf{x}_{3,1}, \mathbf{x}_{3,2}, \ \mathbf{x}_{4,1}\big) \in \mathbb{C}^{n \times 8}$. After we did the whole procedure for all eigenvalues $\lambda_1, \ldots, \lambda_r$, the only thing that remains is:

Put everything together

$$J := \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} \in \mathbb{C}^{n \times n} \quad \textit{and} \quad X := \big(X_1, \ldots, X_r\big) \in \mathbb{C}^{n \times n}. \tag{9.2}$$

This is all. Let us summarise the whole story:

Algorithm for calculating a Jordan normal form of A

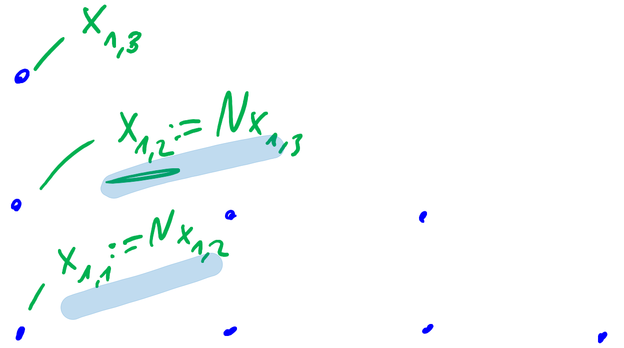
Given: *An arbitrary matrix $A \in \mathbb{C}^{n \times n}$.*
Wanted: *Jordan normal form J and X in $\mathbb{C}^{n \times n}$ with $A = XJX^{-1}$.*

Algorithm

- *Calculate all eigenvalues $\lambda_1, \ldots, \lambda_r$ (pairwise distinct) of A and the algebraic multiplicities $\alpha_1, \ldots, \alpha_r$.*
- *For $i = 1, \ldots, r$:*
  - *Set $N := A - \lambda_i \mathbb{1}$.*
  - *Calculate $\mathrm{Ker}(N), \mathrm{Ker}(N^2), \ldots, \mathrm{Ker}(N^m)$ to $\dim(\cdot) = \alpha_i$.*
  - *Calculate all $d_k := \dim(\mathrm{Ker}(N^k)) - \dim(\mathrm{Ker}(N^{k-1}))$.*
  - *Draw the levels $1, \ldots, m$ and Jordan chains. (Box 9.3)*
  - *Write down the Jordan block J_i. (Box 9.4)*
  - *Calculate all generalised eigenvectors. (Box 9.5)*
  - *Define X_i with all generalised eigenvectors. (Box 9.6)*
- *Set $J := \mathrm{Diag}(J_1, \ldots, J_r)$ and $X := \big(X_1, \ldots, X_r\big)$ as in (9.2).*

Why does this work? Let us look at the X-columns regarding one Jordan chain and its corresponding Jordan box. Choose the first chain from our example.

& gen. eigenvectors



1st level

$$\begin{pmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \boxed{\lambda_i} \end{pmatrix}$$

The chain was given by $\mathbf{x}_{1,2} = N\mathbf{x}_{1,3}$ and $\mathbf{x}_{1,1} = N\mathbf{x}_{1,2}$. In this way, we get.

$$\begin{aligned} & \mathbf{x}_{1,2} = N\mathbf{x}_{1,3} = (A - \lambda_i \mathbb{1})\mathbf{x}_{1,3} = A\mathbf{x}_{1,3} - \lambda_i \mathbf{x}_{1,3}, \quad \text{hence} \quad A\mathbf{x}_{1,3} = \mathbf{x}_{1,2} + \lambda_i \mathbf{x}_{1,3} \\ \text{and} \quad & \mathbf{x}_{1,1} = N\mathbf{x}_{1,2} = (A - \lambda_i \mathbb{1})\mathbf{x}_{1,2} = A\mathbf{x}_{1,2} - \lambda_i \mathbf{x}_{1,2}, \quad \text{hence} \quad A\mathbf{x}_{1,2} = \mathbf{x}_{1,1} + \lambda_i \mathbf{x}_{1,2}. \end{aligned}$$

In summary:

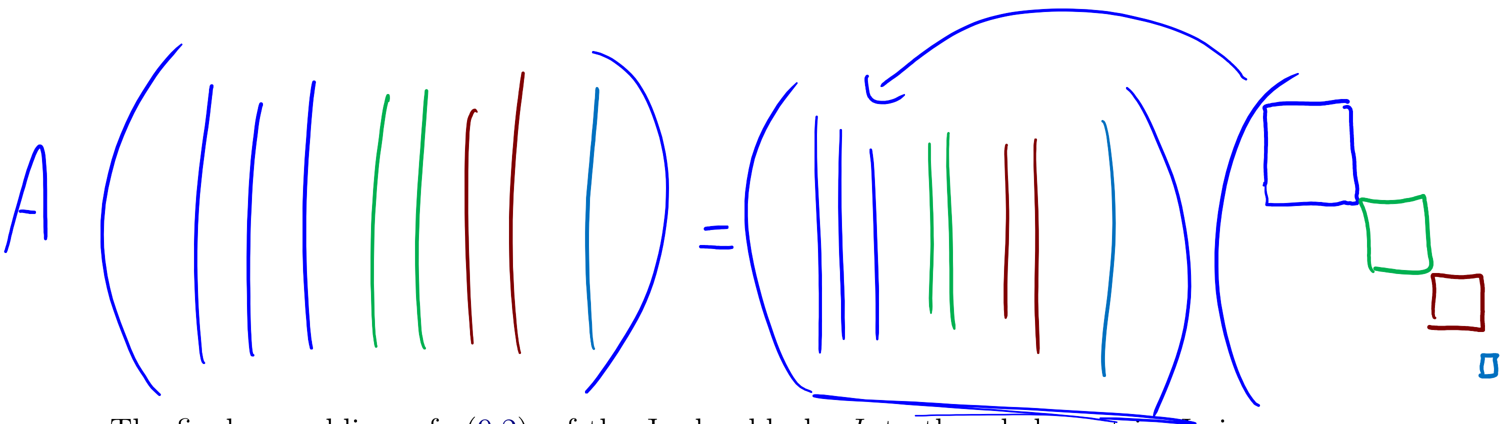
$$\begin{aligned} A \begin{pmatrix} | & | & | \\ \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \\ | & | & | \end{pmatrix} &= \begin{pmatrix} | & | & | \\ A\mathbf{x}_{1,1} & A\mathbf{x}_{1,2} & A\mathbf{x}_{1,3} \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \lambda_i \mathbf{x}_{1,1} & \mathbf{x}_{1,1} + \lambda_i \mathbf{x}_{1,2} & \mathbf{x}_{1,2} + \lambda_i \mathbf{x}_{1,3} \\ | & | & | \end{pmatrix} \\ &= \begin{pmatrix} | & | & | \\ \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \lambda_i \end{pmatrix} =: \begin{pmatrix} | & | & | \\ \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{1,3} \\ | & | & | \end{pmatrix} J_{i,1}. \end{aligned}$$

By using the definition of the Jordan chain, we get the 1s above the diagonal in the matrix $J_{i,1}$. Only at the ordinary eigenvectors (here: $\mathbf{x}_{1,1}$), the chain stops. There, you do not find a 1 but only λ_i since $A\mathbf{x}_{1,1} = \lambda_i \mathbf{x}_{1,1}$.

By putting all Jordan boxes together into a Jordan block, we get γ_i equations (one per Jordan box), given by

$$A\left(\mathbf{x}_{j,1}\ \mathbf{x}_{j,2} \cdots \mathbf{x}_{j,k}\right) = \left(\mathbf{x}_{j,1}\ \mathbf{x}_{j,2} \cdots \mathbf{x}_{j,k}\right) J_{i,j}, \qquad j = 1, \ldots, \gamma_i\,,$$

one matrix equation $AX_i = X_i J_i$ for the ith Jordan block.



The final assembling, cf. (9.2), of the Jordan blocks J_i to the whole matrix J gives us then $AX = XJ$, which is exactly the factorisation $A = XJX^{-1}$.

Now let us practise:

$$\det\begin{pmatrix} 5-\lambda & & 1 \\ & 1-\lambda & \\ -1 & & 3-\lambda \end{pmatrix} \cdot \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$$

Example 9.7. Let

$$A = \begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

$$= [(5-\lambda)(1-\lambda)(3-\lambda) + (1-\lambda)] \cdot [\,\cdot\;\cdot\,]$$

$$= (4-\lambda)^2(1-\lambda) \cdot (1-\lambda)(4-\lambda)$$

- The characteristic polynomial is

$$\det(A - \lambda\mathbb{1}) = (4-\lambda)^3(1-\lambda)^2.$$

We see that $\lambda_1 = 4$ with $\alpha_1 = 3$ and $\lambda_2 = 1$ with $\alpha_2 = 2$.

- Let us start the work (and fun) with the eigenvalue $\lambda_1 = 4$. For the matrix

$$N := A - \lambda_1\mathbb{1} = A - 4\mathbb{1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

free variables

we get (after solving the LES $N\mathbf{x} = \mathbf{o}$) that

$$\mathrm{Ker}(N) = \{\mathbf{x} = (-x_3, 0, x_3, 0, x_5)^\top : x_3, x_5 \in \mathbb{C}\} = \mathrm{span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

and hence $\gamma_1 = \dim(\mathrm{Ker}(N)) = 2$. Since $\alpha_1 = 3$, we have to calculate

$$N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(Square the original one!)

and we get:

free variables

$$\mathrm{Ker}(N^2) = \{\mathbf{x} = (x_1, 0, x_3, 0, x_5)^\top : x_1, x_3, x_5 \in \mathbb{C}\} = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

From this, we conclude $\dim(\mathrm{Ker}(N^2)) = 3$. Now we have reached the algebraic multiplicity $\alpha_1 = 3$ and do not need to consider any higher powers of N, hence $m = 2$.

- For the differences of the dimension, we get

$$d_1 := \dim(\mathrm{Ker}(N^1)) - \dim(\mathrm{Ker}(N^0)) = 2 - 0 = 2,$$
$$d_2 := \dim(\mathrm{Ker}(N^2)) - \dim(\mathrm{Ker}(N^1)) = 3 - 2 = 1.$$

Note that $\mathrm{Ker}(N^0) = \mathrm{Ker}(\mathbb{1}) = \{\mathbf{o}\}$ always have dimension 0.

- We have $m = 2$ levels whereas the second level owns $d_2 = 1$ vectors and the first level has $d_1 = 2$ vectors:

2nd level:	$\mathbf{x}_{1,2}$ $\downarrow$	
1st level:	$\mathbf{x}_{1,1}$	$\mathbf{x}_{2,1}$

- Since we have a Jordan chain with length 2 and another one with length 1, we know that the first Jordan block J_1 has two Jordan blocks with different sizes:

$$J_1 = \begin{pmatrix} \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & \\ & \boxed{4} \end{pmatrix}.$$

- Choose: $x_{1,2} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $x_{1,1} := N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

  eigenvector!

- We have finished the second chain and can give the matrix

$$X_1 = \begin{pmatrix} \mathbf{x}_{1,1} & \mathbf{x}_{1,2} & \mathbf{x}_{2,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

  Now, we have done everything for the eigenvalue $\lambda_1 = 4$. Next thing is the eigenvalue $\lambda_2 = 1$.

- For the matrix

$$N := A - \lambda_2 \mathbb{1} = A - 1\mathbb{1} = \begin{pmatrix} 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

  we get (after solving $N\mathbf{x} = \mathbf{o}$) that

$$\mathrm{Ker}(N) = \{\mathbf{x} = (0, x_2, 0, x_4, 0)^\top : x_2, x_4 \in \mathbb{C}\}$$

  and hence $\gamma_2 = \dim(\mathrm{Ker}(N)) = 2$. Since $\alpha_2 = 2$, we do not need to calculate higher powers of N and set $m = 1$.

- We denote

$$d_1 = \dim(\mathrm{Ker}(N^1)) - \dim(\mathrm{Ker}(N^0)) = 2 - 0 = 2 \ \ldots$$

- ...and get a bit boring picture with only $m = 1$ level and $d_1 = 2$ vectors:

1st level:	$\mathbf{x}_{1,1}$ $\mathbf{x}_{2,1}$

Here, we see two chains with length 1.

- The Jordan block J_2 has two Jordan boxes of size 1 and looks like:

$$J_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

- Let us determine the generalised eigenvectors: $\mathbf{x}_{1,1}$ comes from $\mathrm{Ker}(N^1) \setminus \mathrm{Ker}(N^0)$ and we could choose $\mathbf{x}_{1,1} = (0,1,0,0,0)^\top$. Now, for the second chain, choose $\mathbf{x}_{2,1} \in \mathrm{Ker}(N^1)$ such that is not given by a linear combination of vectors from $\mathrm{Ker}(N^0) \cup \{\mathbf{x}_{1,1}\} = \{\mathbf{o}, \mathbf{x}_{1,1}\}$, cf. (9.1). Let us set $\mathbf{x}_{2,1} = (0,0,0,1,0)^\top$.

- Hence, we have the matrix

$$X_2 = \begin{pmatrix} \mathbf{x}_{1,1} & \mathbf{x}_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and also finished the work for the eigenvalue λ_2.

- In summary, we get:

$$J = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & & & \\ & \boxed{4} & & \\ & & \boxed{1} & \\ & & & \boxed{1} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

hence, $A = XJX^{-1}$.

$A = XJX^{-1}$

Corollary 9.8. Eigenvalues give determinant and trace

For $A \in \mathbb{C}^{n\times n}$, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues counted with algebraic multiplicities. Then

$$\det(A) = \prod_{i=1}^n \lambda_i \quad \text{and} \quad \mathrm{tr}(A) = \sum_{i=1}^n \lambda_i,$$

where $\mathrm{tr}(A) := \sum_{j=1}^n a_{jj}$ is the sum of the diagonal, the so-called trace of A.