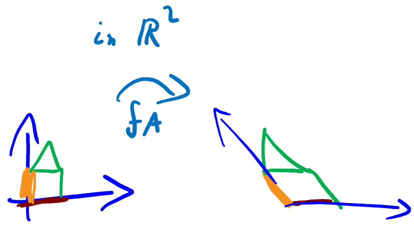
Remark:

A linear map $\ell : V \to W$ exactly conserves the structure of the vector spaces, meaning vector addition and scalar multiplication. Therefore, mathematicians call a linear map a homomorphism. A homomorphism ℓ that is invertible and has an inverse ℓ^{-1} that is also a homomorphism is called an isomorphism.

VL21 ↓

8.3 Finding the matrix for a linear map



8.3.1 Just know what happens to a basis

Rule of thumb: Linearity makes it easy

For a linear map, you only have to know what happens to a basis. The remaining part of space "tags along".

Let $\ell : V \to W$ be a linear map and $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ some basis of V. For each $\mathbf{x} \in V$, we denote by $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n$ its coordinate vector, which means

$$x^{\mathcal{B}} = \Phi_{\mathcal{B}}(\mathbf{x}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n \quad \text{with} \quad \mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n = \Phi_{\mathcal{B}}^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Then:

all we need to know

$$\ell(\mathbf{x}) = \ell(\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n) = \alpha_1 \ell(\mathbf{b}_1) + \cdots + \alpha_n \ell(\mathbf{b}_n)$$

Equation (8.7) says everything: If you know the images of the all basis elements, which means $\ell(\mathbf{b}_1), \dots, \ell(\mathbf{b}_n)$, then you know all images $\ell(\mathbf{x})$ for each $\mathbf{x} \in V$ immediately.

Example 8.13. Let $V = \mathcal{P}_3(\mathbb{R})$ with the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ where $\mathbf{m}_k(x) = x^k$. For the differential operator $\partial \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ where $\partial : \mathbf{f} \mapsto \mathbf{f}'$, we have

$$\partial(\mathbf{m}_0) = \mathbf{o}, \quad \partial(\mathbf{m}_1) = \mathbf{m}_0, \quad \partial(\mathbf{m}_2) = 2\mathbf{m}_1, \quad \partial(\mathbf{m}_3) = 3\mathbf{m}_2, \tag{8.7}$$

$$\downarrow \Phi_{\mathcal{B}} \qquad \downarrow$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$m_3 : x \mapsto x^3$

$m_3' \quad x \mapsto 3x^2$

For an arbitrary $\mathbf{p} \in \mathcal{P}_3(\mathbb{R})$, which means $\mathbf{p}(x) = ax^3 + bx^2 + cx + d$ for $a, b, c, d \in \mathbb{R}$ or $\mathbf{p} = d\mathbf{m}_0 + c\mathbf{m}_1 + b\mathbf{m}_2 + a\mathbf{m}_3$, we have

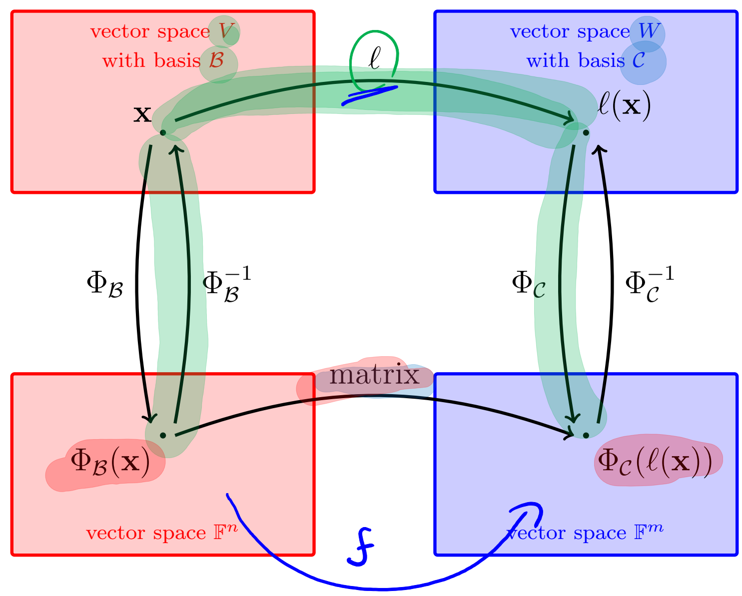
$$\Phi_{\mathcal{B}}(p) = \mathbf{p}^{\mathcal{B}} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \quad \text{and hence } \partial(\mathbf{p}) = d\partial(\mathbf{m}_0) + c\partial(\mathbf{m}_1) + b\partial(\mathbf{m}_2) + a\partial(\mathbf{m}_3) = c\mathbf{m}_0 + 2b\mathbf{m}_1 + 3a\mathbf{m}_2.$$

$$\xmapsto{\Phi_{\mathcal{B}}} \begin{pmatrix} c \\ 2b \\ 3a \\ 0 \end{pmatrix}$$

Checking this: $\mathbf{p}'(x) = 3ax^2 + 2bx + c$, hence $\partial(\mathbf{p}) = \mathbf{p}' = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0$.

8.3.2 Matrix of a linear map with respect to bases

Let us consider again two arbitrary finite-dimensional $\mathbb{F}$-vector spaces V and W and linear maps between them.



$f_A = \Phi_\mathcal{C} \circ \ell \circ \Phi_\mathcal{B}^{-1}$
linear

$f_A : \mathbb{F}^n \to \mathbb{F}^m$
$\Updownarrow$
$A = \begin{pmatrix} | & & | \\ f_A(e_1) & \cdots & f_A(e_n) \\ | & & | \end{pmatrix}$

$f_A(x) = Ax$

Question:

How to get the map or the matrix in the bottom. How to send the coordinate vector $\Phi_\mathcal{B}(\mathbf{x})$ to the coordinate vector $\Phi_\mathcal{C}(\ell(\mathbf{x}))$?

Of course, this is given by composing the three maps:

$$\Phi_\mathcal{C}(\ell(\mathbf{x})) = (\Phi_\mathcal{C} \circ \ell \circ \Phi_\mathcal{B}^{-1})(\Phi_\mathcal{B}(\mathbf{x}))$$

$$j\text{th column of } A = f_A(e_j) = (\Phi_\mathcal{C} \circ \ell \circ \Phi_\mathcal{B}^{-1})(\mathbf{e}_j) = \Phi_\mathcal{C}(\ell(\underbrace{\Phi_\mathcal{B}^{-1}(\mathbf{e}_j)}_{=b_j})) = \Phi_\mathcal{C}(\ell(\mathbf{b}_j))$$

This gives us a matrix that really represents the abstract linear map. It depends, of course, on the chosen bases $\mathcal{B}$ and $\mathcal{C}$ in the vector spaces V and W, respectively. Therefore, we choose a good name:

Matrix representation of the linear map

For the linear map $\ell : V \to W$, we define the matrix

$$\ell_{\mathcal{C}\leftarrow\mathcal{B}} := \begin{pmatrix} | & & | \\ \Phi_\mathcal{C}(\ell(\mathbf{b}_1)) & \cdots & \Phi_\mathcal{C}(\ell(\mathbf{b}_n)) \\ | & & | \end{pmatrix} \in \mathbb{F}^{m\times n} \tag{8.8}$$

and call it the matrix representation of the linear map ℓ with respect to the basis $\mathcal{B}$ and $\mathcal{C}$.

All the information of ℓ is in this matrix
(You need to know the bases)

This gets us to:

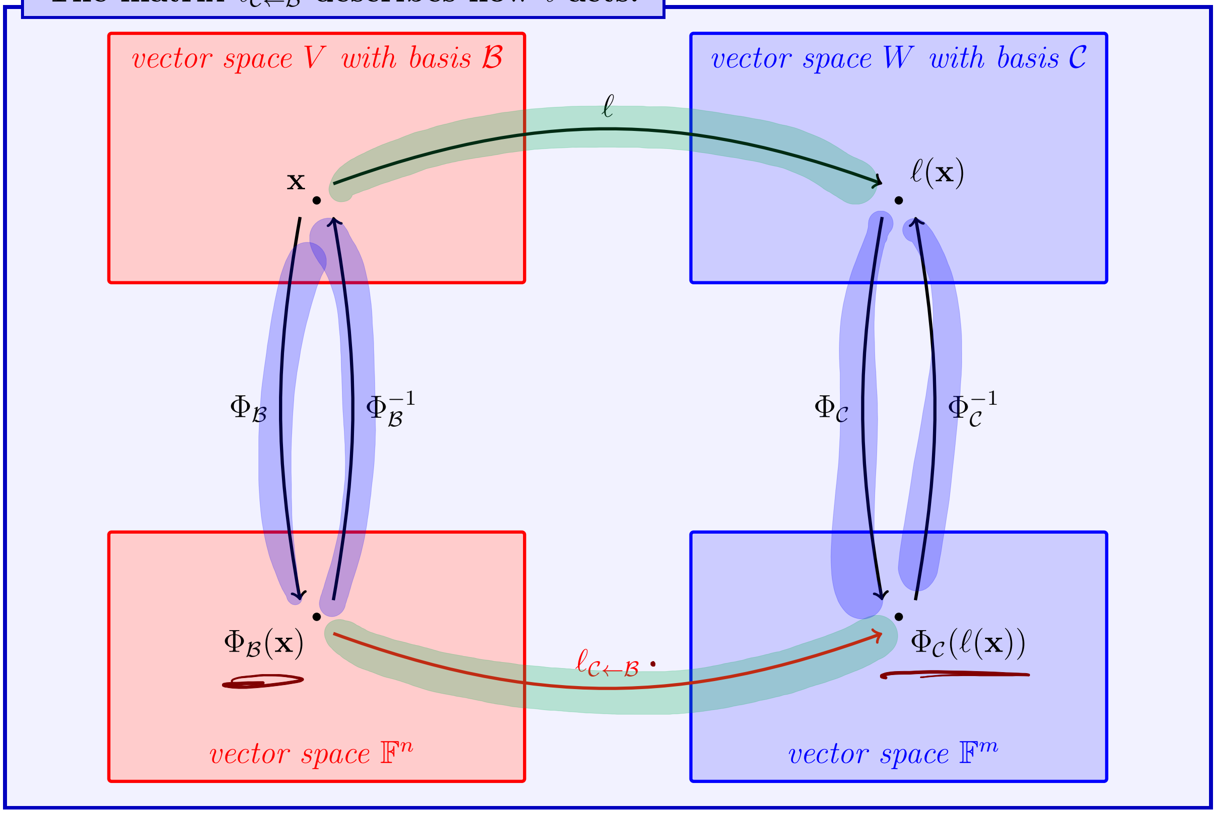
In short: $\ell(x)^{\mathcal{C}} = \ell_{\mathcal{C}\leftarrow\mathcal{B}}\, x^{\mathcal{B}}$

How to map the coordinates

$$\Phi_{\mathcal{C}}(\ell(\mathbf{x})) = \ell_{\mathcal{C}\leftarrow\mathcal{B}}\Phi_{\mathcal{B}}(\mathbf{x}). \tag{8.9}$$

This completes our picture:

The matrix $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$ describes how ℓ acts:



Example 8.14. (a) Let $\partial : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ with $\mathbf{f} \mapsto \mathbf{f}'$ the differential operator We use in $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ the respective monomial basis:

$$\mathcal{B} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) \quad \text{and} \quad \mathcal{C} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0).$$

We already know:

$\in \mathbb{R}^3$

$$\Phi_{\mathcal{C}}(\partial(\mathbf{m}_3)) = \Phi_{\mathcal{C}}(3\mathbf{m}_2) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\begin{matrix} m_2 \\ m_1 \\ m_0 \end{matrix}, \qquad \Phi_{\mathcal{C}}(\partial(\mathbf{m}_2)) = \Phi_{\mathcal{C}}(2\mathbf{m}_1) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix},$$

$$\Phi_{\mathcal{C}}(\partial(\mathbf{m}_1)) = \Phi_{\mathcal{C}}(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \Phi_{\mathcal{C}}(\partial(\mathbf{m}_0)) = \Phi_{\mathcal{C}}(\mathbf{o}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3.$$

The column vectors from above give us the columns of the matrix $\partial_{\mathcal{C}\leftarrow\mathcal{B}}$:

$$\partial_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | & | & | \\ \Phi_{\mathcal{C}}(\partial(\mathbf{m}_3)) & \Phi_{\mathcal{C}}(\partial(\mathbf{m}_2)) & \Phi_{\mathcal{C}}(\partial(\mathbf{m}_1)) & \Phi_{\mathcal{C}}(\partial(\mathbf{m}_0)) \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{8.10}$$

Now we can use the map ∂ just on the coordinate level: For $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ given by $\mathbf{f}(x) = ax^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{R}$, we have

$f'(x) = 3ax^2 + 2bx + c$

$$\begin{tikzcd} \mathcal{P}_3(\mathbb{R}) \arrow[r, "\partial"] & \mathcal{P}_2(\mathbb{R}) \\ \mathbb{R}^4 \arrow[u] \arrow[r, "\partial_{\mathcal{C}\leftarrow\mathcal{B}}"] & \mathbb{R}^3 \arrow[u] \end{tikzcd}$$

$$\Phi_{\mathcal{B}}(\mathbf{f}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \text{hence} \quad \Phi_{\mathcal{B}}(\partial(\mathbf{f})) = \partial_{\mathcal{C}\leftarrow\mathcal{B}}\, \Phi_{\mathcal{B}}(\mathbf{f}) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3a \\ 2b \\ c \end{pmatrix}.$$

So we get:

$$\partial(\mathbf{f}) = \Phi_{\mathcal{C}}^{-1} \begin{pmatrix} 3a \\ 2b \\ c \end{pmatrix} = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0.$$

We check this again by $\partial(\mathbf{f}) = \mathbf{f}'$ and $\mathbf{f}'(x) = 3ax^2 + 2bx + c$ for all x. Therefore, $\partial(\mathbf{f}) = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0$. Great!

(b) Looking again at the map $\int : \mathcal{P}_2([0,1]) \to \mathcal{P}_3([0,1])$ which sends $\mathbf{f}$ to its antiderivative $\mathbf{F}$ given by

↳ Homework

$$\mathbf{F}(x) = \int_0^x \mathbf{f}(t)\,dt \qquad \text{for all} \quad x \in [0,1].$$

Take again the monomial basis $\mathcal{B} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ for $\mathcal{P}_2([0,1])$ and $\mathcal{C} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ for $\mathcal{P}_3([0,1])$. For getting the matrix $\int_{\mathcal{C}\leftarrow\mathcal{B}}$, we need the images of $\mathcal{B}$. Because of

$$\int(\mathbf{m}_k)(x) = \int_0^x t^k\,dt = \frac{t^{k+1}}{k+1}\Big|_0^x = \frac{x^{k+1}}{k+1} = \frac{1}{k+1}\mathbf{m}_{k+1}(x) \qquad \text{for} \qquad k = 2,1,0\,,$$

we get

$$\Phi_{\mathcal{C}}(\textstyle\int(\mathbf{m}_2)) = \Phi_{\mathcal{C}}(\frac{1}{3}\mathbf{m}_3) = \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix},$$

$$\Phi_{\mathcal{C}}(\textstyle\int(\mathbf{m}_1)) = \Phi_{\mathcal{C}}(\frac{1}{2}\mathbf{m}_2) = \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix},$$

$$\Phi_{\mathcal{C}}(\textstyle\int(\mathbf{m}_0)) = \Phi_{\mathcal{C}}(\frac{1}{1}\mathbf{m}_1) = \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix}.$$

The matrix representation $\int_{\mathcal{C}\leftarrow\mathcal{B}}$ is now given by the coordinate vectors with respect to the basis $\mathcal{C}$:

$$\int_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | & | \\ \Phi_{\mathcal{C}}(\int(\mathbf{m}_2)) & \Phi_{\mathcal{C}}(\int(\mathbf{m}_1)) & \Phi_{\mathcal{C}}(\int(\mathbf{m}_0)) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (8.11)$$

$$\begin{tikzcd} V=\mathbb{F}^n \arrow[r, "f_A"', "A\cdot"] \arrow[d] & V=\mathbb{F}^m \arrow[d] \\ \mathbb{F}^n \arrow[r, "(f_A)_{\mathcal{C}\leftarrow\mathcal{B}}", "A\cdot"'] & \mathbb{F}^m \end{tikzcd}$$

(c) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Choose

$$A = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \in \mathbb{F}^{m\times n}$$

and the associated linear map $f_A : \mathbb{F}^n \to \mathbb{F}^m$ with $f_A : \mathbf{x} \mapsto A\mathbf{x}$. For a basis in $V = \mathbb{F}^n$, we choose $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ and in $W = \mathbb{F}^m$ canonical basis $\mathcal{C} = (\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_m)$, where we choose the hats just to distinguish this basis from $\mathcal{B}$. For getting the matrix representation $(f_A)_{\mathcal{C}\leftarrow\mathcal{B}}$ we look what f_A does with the basis $\mathcal{B}$:

$$\underbrace{\Phi_{\mathcal{C}}}_{id}(f_A(\underbrace{b_1}_{e_1}))$$

$$= f_A(\mathbf{e}_1) = A\mathbf{e}_1 = \mathbf{a}_1 \overset{(*)}{=} \Phi_{\mathcal{C}}^{-1}\mathbf{a}_1, \quad \dots, \quad f_A(\mathbf{e}_n) = A\mathbf{e}_n = \mathbf{a}_n \overset{(*)}{=} \Phi_{\mathcal{C}}^{-1}\mathbf{a}_n$$

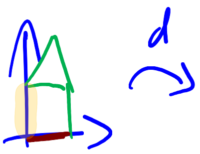
$$\left(\begin{matrix} \Phi_{\mathcal{B}} = id \\ \Phi_{\mathcal{C}} = id \end{matrix} \right)$$

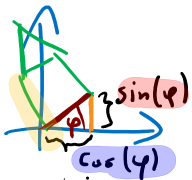
For the matrix representation $(f_A)_{\mathcal{C}\leftarrow\mathcal{B}}$, we write the images into the columns and get:

$$(f_A)_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} = A.$$

[matrix representation of f_A w.r.t. standard bases is A !]

(d) Let $d : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by angle φ. Choose in $V = W = \mathbb{R}^2$ the canonical basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$. We use the rotation d for the basis elements $\mathbf{e}_1 = \binom{1}{0}$ and $\mathbf{e}_2 = \binom{0}{1}$:





$$d(\mathbf{e}_1) = d\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} = \Phi_{\mathcal{B}}^{-1}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix},$$

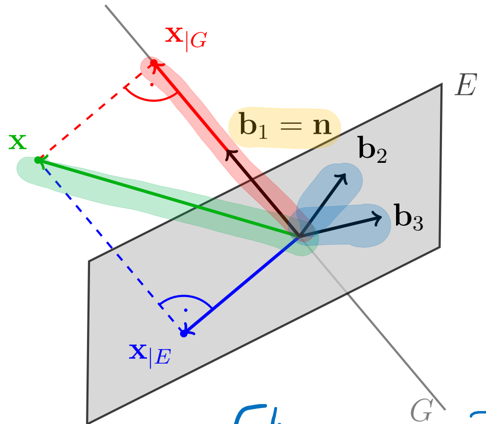
$$d(\mathbf{e}_2) = d\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} = \Phi_{\mathcal{B}}^{-1}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}.$$

The matrix representation of d with respect to the standard basis is a so-called rotation matrix

"Rotation matrix" = matrix representation of rotation with φ

$$d_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}. \tag{8.12}$$

(e) Let $\mathbf{n} \in \mathbb{R}^3$ with $\|\mathbf{n}\| = 1$ and $\text{proj}_G : \mathbb{R}^3 \to \mathbb{R}^3$ the linear map given by the orthogonal projection onto $G := \text{Span}(\mathbf{n})$. We choose a basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, in both basis $\mathbb{R}^3$, which fits our problem: Let $\mathbf{b}_1 := \mathbf{n}$ and $\mathbf{b}_2$ and $\mathbf{b}_3$ orthogonal to $\mathbf{n}$. Then:



$$\text{proj}_G : \quad \mathbf{x} = \underbrace{\alpha\mathbf{b}_1}_{\mathbf{x}_{|G}} + \underbrace{\beta\mathbf{b}_2 + \gamma\mathbf{b}_3}_{\mathbf{x}_{|E}} \quad \mapsto \quad \underbrace{\alpha\mathbf{b}_1}_{\mathbf{x}_{|G}}$$

or in the coordinate language:

$$(\text{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} : \quad \Phi_{\mathcal{B}}(\mathbf{x}) = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \overset{\text{linear map}}{\mapsto} \Phi_{\mathcal{B}}(\mathbf{x}_{|G}) = \begin{pmatrix}\alpha\\ 0\\ 0\end{pmatrix}.$$

There, we can immediately see the matrix representation $(\text{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}}$:

[because we chose a good basis for our problem]

$$(\text{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\\ 0\end{pmatrix} \tag{8.13}$$

Alternatively, you would calculate the images:

$$\Phi_{\mathcal{B}}(\mathrm{proj}_G(\mathbf{b}_1)) = \Phi_{\mathcal{B}}(\mathbf{b}_1) = \begin{array}{cc} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \end{array}$$

$$\Phi_{\mathcal{B}}(\mathrm{proj}_G(\mathbf{b}_2)) = \Phi_{\mathcal{B}}(\mathbf{o}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Phi_{\mathcal{B}}(\mathrm{proj}_G(\mathbf{b}_3)) = \Phi_{\mathcal{B}}(\mathbf{o}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

8.3.3 Matrix representation for compositions

Linear maps: $+$, $\alpha\cdot$, $\circ$

Proposition 8.15. Operations for matrix represenations

(a) *Let V and W be two $\mathbb{F}$-vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively. For linear maps $k, \ell \in \mathcal{L}(V, W)$ and $\alpha \in \mathbb{F}$, we have*

$$(k + \ell)_{\mathcal{C}\leftarrow\mathcal{B}} = k_{\mathcal{C}\leftarrow\mathcal{B}} + \ell_{\mathcal{C}\leftarrow\mathcal{B}} \qquad \text{and} \qquad (\alpha\,\ell)_{\mathcal{C}\leftarrow\mathcal{B}} = \alpha\,\ell_{\mathcal{C}\leftarrow\mathcal{B}}.$$

(b) *Let U be a third $\mathbb{F}$-vector space with chosen basis $\mathcal{A}$. For all $\ell \in \mathcal{L}(U, V)$ and $k \in \mathcal{L}(V, W)$, we have*

$$(k \circ \ell)_{\mathcal{C}\leftarrow\mathcal{A}} = k_{\mathcal{C}\leftarrow\mathcal{B}}\,\ell_{\mathcal{B}\leftarrow\mathcal{A}}.$$

$$\begin{tikzcd} U \arrow[r, "\ell"] \arrow[rr, bend left] & V \arrow[r, "k"] & W \\ \mathbb{F}^n \arrow[r, "\ell_{\mathcal{B}\leftarrow\mathcal{A}}"] & \mathbb{F}^k \arrow[r, "k_{\mathcal{C}\leftarrow\mathcal{B}}"] & \mathbb{F}^m \end{tikzcd}$$

The zero matrix 0 and the identity matrix $\mathbb{1}$ are exactly the matrix representations of the zero map $o : V \to W$ with $\mathbf{x} \mapsto \mathbf{o}$ and of the identity map $id : V \to V$ with $\mathbf{x} \mapsto \mathbf{x}$, respectively.

$$o_{\mathcal{C}\leftarrow\mathcal{B}} = 0 \qquad \text{and} \qquad id_{\mathcal{B}\leftarrow\mathcal{B}} = \mathbb{1}.$$

Now choose ℓ again as a linear map $V \to W$ and also a basis $\mathcal{B}$ in V and a basis $\mathcal{C}$ in W. If ℓ is invertible, we immediately get:

$$(\ell^{-1})_{\mathcal{B}\leftarrow\mathcal{C}}\,\ell_{\mathcal{C}\leftarrow\mathcal{B}} = (\ell^{-1} \circ \ell)_{\mathcal{B}\leftarrow\mathcal{B}} = id_{\mathcal{B}\leftarrow\mathcal{B}} = \mathbb{1}_n \qquad \text{and} \qquad \ell_{\mathcal{C}\leftarrow\mathcal{B}}\,(\ell^{-1})_{\mathcal{B}\leftarrow\mathcal{C}} = \mathbb{1}_m$$

$\dim(V) = n$ $\qquad$ $\dim(W) = m$

Hence:

$A \cdot B = \mathbb{1}_n$ $\qquad$ $\boxed{\text{Homework 6.3}}$ $\qquad$ $B \cdot A = \mathbb{1}_m$

Matrix representation of inverse = inverse matrix

$$(\ell^{-1})_{\mathcal{B}\leftarrow\mathcal{C}} = (\ell_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}. \tag{8.14}$$

From this, we can conclude a very important result:

Corollary 8.16. Bijectivity not possible, if $\dim(V) \neq \dim(W)$

If $\dim(V) \neq \dim(W)$, then all linear maps $\ell : V \to W$ are not invertible.

For $\dim(V) \neq \dim(W)$, you can still have bijective maps! (but no linear ones)

Proof. If ℓ is invertible, then (8.14) says the $m \times n$-matrix $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. This means that the matrix is a square one, hence $\dim(V) = n = m = \dim(W)$. □

$$(\mathrm{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 8.17. (a) Let $\mathrm{proj}_G \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be the linear operator given by the orthogonal projection onto $G := \mathrm{Span}(\mathbf{n})$. We choose the same basis $\mathcal{B}$ in both $\mathbb{R}^3$ like in Example 8.14 (f). For the projection proj_E and the reflection refl_E with respect to the plane $E := \{\mathbf{n}\}^\perp$, Proposition 8.15 gives us:

$$\begin{aligned}(\mathrm{proj}_E)_{\mathcal{B}\leftarrow\mathcal{B}} &\overset{(8.2)}{=} (id - \mathrm{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} \\ &= id_{\mathcal{B}\leftarrow\mathcal{B}} - (\mathrm{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} \overset{(8.13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

"reflection"

$$\begin{aligned}(\mathrm{refl}_E)_{\mathcal{B}\leftarrow\mathcal{B}} &\overset{(8.2)}{=} (id - 2\,\mathrm{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} \\ &= id_{\mathcal{B}\leftarrow\mathcal{B}} - 2\,(\mathrm{proj}_G)_{\mathcal{B}\leftarrow\mathcal{B}} \overset{(8.13)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

(b) Next, we again consider the differential operator $\partial : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ and the anti-derivative operator $\int : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$. In $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$ choose the monomial basis $\mathcal{B}$ and $\mathcal{C}$, respectively. From Proposition 8.15 and the equations (8.10) and (8.11), we conclude

$$(\partial \circ \textstyle\int)_{\mathcal{B}\leftarrow\mathcal{B}} = \partial_{\mathcal{B}\leftarrow\mathcal{C}} \ \textstyle\int_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = id_{\mathcal{B}\leftarrow\mathcal{B}}$$

and

$$(\textstyle\int \circ \partial)_{\mathcal{C}\leftarrow\mathcal{C}} = \textstyle\int_{\mathcal{C}\leftarrow\mathcal{B}} \ \partial_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq id_{\mathcal{C}\leftarrow\mathcal{C}}.$$

8.3.4 Change of basis

$id : V \to V \,, \quad \Phi_{\mathcal{C}}(id(b_1)) = b_1^{\mathcal{C}}$

Let $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ be two bases of V. Then, the identity map $id : \mathbf{x} \mapsto \mathbf{x}$ of V with respect to $\mathcal{B}$ and $\mathcal{C}$ has the following matrix representation:

$$id_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & & | \\ \Phi_{\mathcal{C}}(id(\mathbf{b}_1)) & \dots & \Phi_{\mathcal{C}}(id(\mathbf{b}_n)) \\ | & & | \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^{\mathcal{C}} & \cdots & \mathbf{b}_n^{\mathcal{C}} \end{pmatrix} = T_{\mathcal{C}\leftarrow\mathcal{B}}. \tag{8.15}$$

Choose a good basis for your problem!

Now: $\ell : V \to W \rightsquigarrow \ell_{\mathcal{C}\leftarrow\mathcal{B}}$ matrix w.r.t. $\mathcal{B}$ and $\mathcal{C}$

$\rightsquigarrow \ell_{\mathcal{C}'\leftarrow\mathcal{B}'}$ matrix w.r.t. $\mathcal{B}'$ and $\mathcal{C}'$

Question:

What is the relation between $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$ and $\ell_{\mathcal{C}'\leftarrow\mathcal{B}'}$?

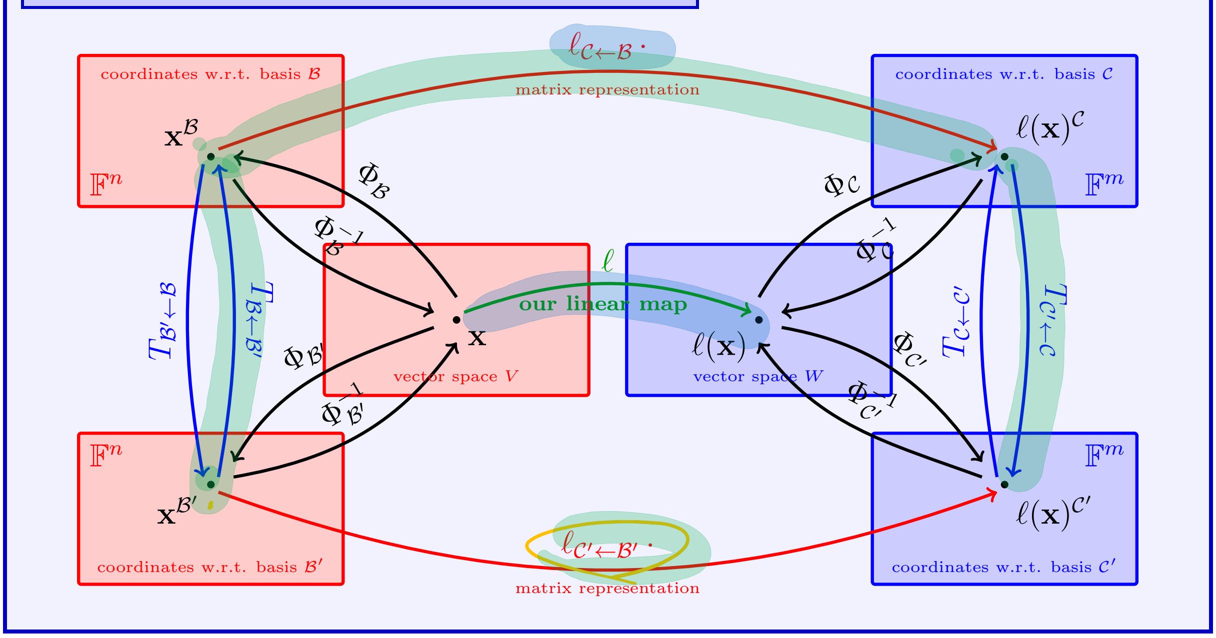
Let us try to calculate the matrices $\ell_{\mathcal{C}'\leftarrow\mathcal{B}'}$ with the help of $\ell_{\mathcal{C}\leftarrow\mathcal{B}}$:

Change of basis left and right

$$\ell_{\mathcal{C}'\leftarrow\mathcal{B}'} = (id \circ \ell \circ id)_{\mathcal{C}'\leftarrow\mathcal{B}'} = id_{\mathcal{C}'\leftarrow\mathcal{C}}\, \ell_{\mathcal{C}\leftarrow\mathcal{B}}\, id_{\mathcal{B}\leftarrow\mathcal{B}'} = T_{\mathcal{C}'\leftarrow\mathcal{C}}\, \ell_{\mathcal{C}\leftarrow\mathcal{B}}\, T_{\mathcal{B}\leftarrow\mathcal{B}'} \tag{8.16}$$

This gives us a nice picture:

Diagram: Change of basis left and right



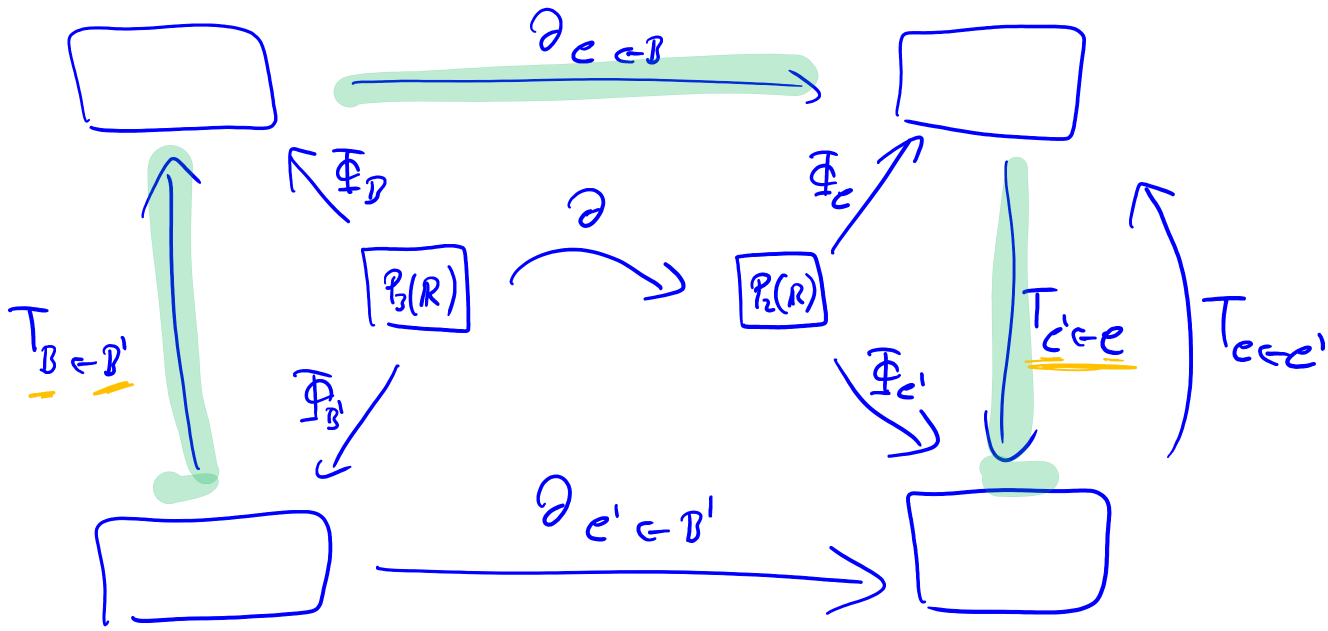
Example 8.18. Let us consider the differential operator $\partial : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ where $V = \mathcal{P}_3(\mathbb{R})$ carries the monomial basis $\mathcal{B} = (\mathbf{m}_3, \mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ and an additional basis

$$\mathcal{B}' = (2\mathbf{m}_3 - \mathbf{m}_1,\ \mathbf{m}_2 + \mathbf{m}_0,\ \mathbf{m}_1 + \mathbf{m}_0,\ \mathbf{m}_1 - \mathbf{m}_0) =: (\mathbf{b}'_1,\ \mathbf{b}'_2,\ \mathbf{b}'_3,\ \mathbf{b}'_4)\,.$$

Moreover, $W = \mathcal{P}_2(\mathbb{R})$ carries the monomial basis $\mathcal{C} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0)$ and another basis

$$\mathcal{C}' = (\mathbf{m}_2 - \tfrac{1}{2}\mathbf{m}_1,\ \mathbf{m}_2 + \tfrac{1}{2}\mathbf{m}_1,\ \mathbf{m}_0) =: (\mathbf{c}'_1,\ \mathbf{c}'_2,\ \mathbf{c}'_3).$$

$$\partial_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$\Phi_{\mathcal{B}}(\mathbf{b}'_1) = \Phi_{\mathcal{B}}(2\mathbf{m}_3 - \mathbf{m}_1) = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \Phi_{\mathcal{B}}(\mathbf{b}'_2) = \Phi_{\mathcal{B}}(\mathbf{m}_2 + \mathbf{m}_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\Phi_{\mathcal{B}}(\mathbf{b}'_3) = \Phi_{\mathcal{B}}(\mathbf{m}_1 + \mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \Phi_{\mathcal{B}}(\mathbf{b}'_2) = \Phi_{\mathcal{B}}(\mathbf{m}_1 - \mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

In summary, we have:

$$T_{\mathcal{C}'\leftarrow\mathcal{C}} = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \partial_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad T_{\mathcal{B}\leftarrow\mathcal{B}'} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

Using (8.16), we know that the matrix representation $\partial_{\mathcal{C}'\leftarrow\mathcal{B}'}$ is given by the product of these three matrices:

$$\partial_{\mathcal{C}'\leftarrow\mathcal{B}'} = T_{\mathcal{C}'\leftarrow\mathcal{C}}\, \partial_{\mathcal{C}\leftarrow\mathcal{B}}\, T_{\mathcal{B}\leftarrow\mathcal{B}'} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}. \tag{8.17}$$

Alternatively, we could directly calculate $\partial_{\mathcal{C}'\leftarrow\mathcal{B}'}$ from ∂ and the bases $\mathcal{B}'$ and $\mathcal{C}'$. In order to do this, we apply ∂ to the basis elements from $\mathcal{B}'$ and represent the results with respect to the basis $\mathcal{C}'$:

$$\Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_1)) = \Phi_{\mathcal{C}'}\Big(\partial(\underbrace{2\mathbf{m}_3 - \mathbf{m}_1}_{\mathbf{b}'_1})\Big) = \Phi_{\mathcal{C}'}(6\mathbf{m}_2 - \mathbf{m}_0) = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix},$$

$$\Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_2)) = \Phi_{\mathcal{C}'}\Big(\partial(\underbrace{\mathbf{m}_2 + \mathbf{m}_0}_{\mathbf{b}'_2})\Big) = \Phi_{\mathcal{C}'}(2\mathbf{m}_1) = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix},$$

$$\Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_3)) = \Phi_{\mathcal{C}'}\Big(\partial(\underbrace{\mathbf{m}_1 + \mathbf{m}_0}_{\mathbf{b}'_3})\Big) = \Phi_{\mathcal{C}'}(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\Phi_{\mathcal{C}'}(\partial(\mathbf{b}'_4)) = \Phi_{\mathcal{C}'}\Big(\partial(\underbrace{\mathbf{m}_1 - \mathbf{m}_0}_{\mathbf{b}'_4})\Big) = \Phi_{\mathcal{C}'}(\mathbf{m}_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This gives us, as expected, the same matrix as in (8.17).
However, we can also do another alternative computation. Choose $a, b, c, d \in \mathbb{R}$ arbitrarily. Then:

$$\begin{aligned}
\mathbf{f}^{\mathcal{B}'} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad &\xmapsto{\Phi^{-1}_{\mathcal{B}'}} \quad \mathbf{f} = a(2\mathbf{m}_3 - \mathbf{m}_1) + b(\mathbf{m}_2 + \mathbf{m}_0) + c(\mathbf{m}_1 + \mathbf{m}_0) + d(\mathbf{m}_1 - \mathbf{m}_0) \\
&\qquad\quad = 2a\mathbf{m}_3 + b\mathbf{m}_2 + (-a + c + d)\mathbf{m}_1 + (b + c - d)\mathbf{m}_0 \\
&\xmapsto{\partial} \quad \partial(\mathbf{f}) = 6a\mathbf{m}_2 + 2b\mathbf{m}_1 + (-a + c + d)\mathbf{m}_0 \\
&\qquad\quad = 6a\underbrace{\left(\tfrac{1}{2}\mathbf{c}'_1 + \tfrac{1}{2}\mathbf{c}'_2\right)}_{\mathbf{m}_2} + 2b\underbrace{\left(-\mathbf{c}'_1 + \mathbf{c}'_2\right)}_{\mathbf{m}_1} + (-a + c + d)\underbrace{\mathbf{c}'_3}_{\mathbf{m}_0} \\
&\qquad\quad = (3a - 2b)\mathbf{c}'_1 + (3a + 2b)\mathbf{c}'_2 + (-a + c + d)\mathbf{c}'_3 \\
&\xmapsto{\Phi_{\mathcal{C}'}} \quad \partial(\mathbf{f})^{\mathcal{C}'} = \begin{pmatrix} 3a - 2b \\ 3a + 2b \\ -a + c + d \end{pmatrix} \overset{\checkmark}{=} \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.
\end{aligned}$$

8.3.5 Equivalent and similar matrices

Both matrices

$$\partial_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \partial_{\mathcal{C}' \leftarrow \mathcal{B}'} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

from Example 8.18 look completely different although they describe the same linear map $\partial \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$, however, with respect two different bases.

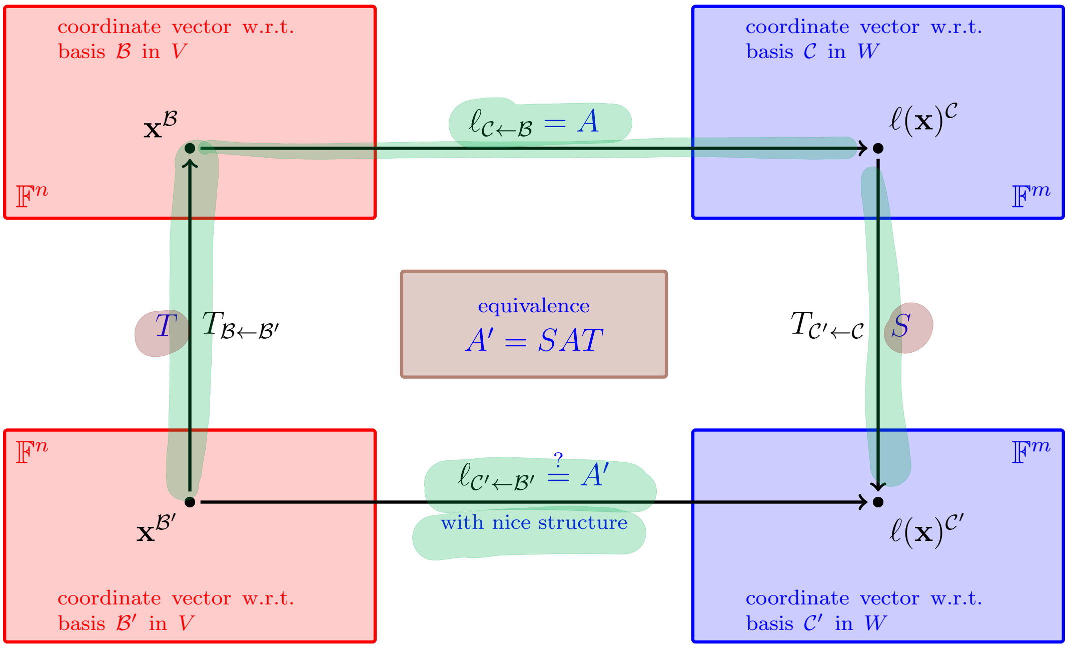
In general: $\ell \in \mathcal{L}(V, W)$, $A := \ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}$

Another matrix $A' \in \mathbb{F}^{m \times n}$
(simpler?)

Question:

Are there bases $\mathcal{B}'$ and $\mathcal{C}'$ in V and W, respectively, such that A' is the matrix representation of ℓ,

$$A' = \ell_{\mathcal{C}'\leftarrow\mathcal{B}'} ?$$



We already know, cf. (8.16),

$$\ell_{\mathcal{C}'\leftarrow\mathcal{B}'} = \underbrace{T_{\mathcal{C}'\leftarrow\mathcal{C}}}_{=:S}\underbrace{\ell_{\mathcal{C}\leftarrow\mathcal{B}}}_{A}\underbrace{T_{\mathcal{B}\leftarrow\mathcal{B}'}}_{=:T} = SAT.$$

Choosing all possible bases $\mathcal{B}'$ and $\mathcal{C}'$ in V and W, respectively, we get all possible invertible matrices S and T and hence with $\ell_{\mathcal{C}'\leftarrow\mathcal{B}'}$ all matrices that are equivalent to A:

Proposition & Definition 8.19. Equivalent matrices

A matrix $B \in \mathbb{F}^{m\times n}$ is called equivalent to another matrix $A \in \mathbb{F}^{m\times n}$ if there are invertible matrices $S \in \mathbb{F}^{m\times m}$ and $T \in \mathbb{F}^{n\times n}$ with

$$B = SAT.$$

In this case, we write $B \sim A$. For arbitrary matrices $A, B, C \in \mathbb{F}^{m\times n}$, the following holds:

$$A \sim A, \qquad A \sim B \Rightarrow B \sim A, \qquad A \sim B \wedge B \sim C \Rightarrow A \sim C.$$

Same properties as for $=$
as for $\Leftrightarrow$ in logic

Equivalent matrices describe the same linear map, just with respect to different bases.