Proposition & Definition 7.39. Still the same about orthogonality:

- *For* $\mathbf{x}, \mathbf{y} \in V$ *we write* $\mathbf{x} \perp \mathbf{y}$ *if* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- *For* $\mathbb{F} = \mathbb{R}$ *and* $\mathbf{x}, \mathbf{y} \in V \setminus \{\mathbf{o}\}$ *we define:*

$$\operatorname{angle}(\mathbf{x}, \mathbf{y}) \;:=\; \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}\right).$$

- *For a nonempty set* $M \subset V$ *we call*

$$M^{\perp} \;:=\; \{\mathbf{x} \in V : \mathbf{x} \perp \mathbf{m} \textit{ for all } \mathbf{m} \in M\}$$

the orthogonal complement of M*. This is always a subspace of* V*.*
Instead of $\mathbf{x} \in M^{\perp}$*, we often write* $x \perp M$*.*

- *For* $\mathbf{x} \in V$ *and a subspace* U *of* V *there is a unique decomposition*

$$\mathbf{x} \;=\; \mathbf{p} + \mathbf{n} \;=:\; \mathbf{x}_{|U} + \mathbf{x}_{|U^{\perp}}$$

into the orthogonal projection $\mathbf{p} =: \mathbf{x}_{|U} \in U$ *and the normal component* $\mathbf{n} = \mathbf{x}_{|U^{\perp}} \in U^{\perp}$ *with respect to* U*. The calculation is given by*

$$G(\mathcal{B})\, \Phi_{\mathcal{B}}(\mathbf{p}) = \begin{pmatrix} \langle \mathbf{x}, \mathbf{b}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{b}_n \rangle \end{pmatrix} \tag{7.21}$$

for any basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ *of* U*, and* $\mathbf{n} = \mathbf{x} - \mathbf{p}$.

- *A family* $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ *with vectors from* V *is called:*
  - *Orthogonal system (OS) if* $\mathbf{u}_i \perp \mathbf{u}_j$ *for all* $i, j = 1, \ldots, n$ *with* $i \neq j$*;*
  - *Orthonormal system (ONS) if, in addition,* $\|\mathbf{u}_i\| = 1$ *for all* $i = 1, \ldots, n$*;*
  - *Orthogonal basis (OB) if it an OS and a basis of* V*;*
  - *Orthonormal basis (ONB) if it an ONS and a basis of* V*.*
- *OS that do not own the zero vector* $\mathbf{o}$ *are always linearly independent.*
- *If* $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ *is an OB of* U*, then the equation (7.21) is much simpler:*

$$\Phi_{\mathcal{B}}(\mathbf{x}_{|U}) = \begin{pmatrix} \frac{\langle \mathbf{x}, \mathbf{b}_1 \rangle}{\|\mathbf{b}_1\|^2} \\ \vdots \\ \frac{\langle \mathbf{x}, \mathbf{b}_n \rangle}{\|\mathbf{b}_n\|^2} \end{pmatrix}, \quad \textit{i.e.} \quad \mathbf{x}_{|U} = \frac{\langle \mathbf{x}, \mathbf{b}_1 \rangle}{\|\mathbf{b}_1\|^2}\mathbf{b}_1 + \ldots + \frac{\langle \mathbf{x}, \mathbf{b}_n \rangle}{\|\mathbf{b}_n\|^2}\mathbf{b}_n. \tag{7.22}$$

If $\mathcal{B}$ *is an ONB, then it gets also easier* $\|\mathbf{b}_i\|^2$ $(= 1)$.

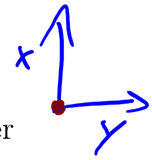
VL20 ↓

Example 7.40. (a) The vectors $\mathbf{x} = \binom{1}{\mathrm{i}}$ and $\mathbf{y} = \binom{0}{1}$ from $\mathbb{C}^2$ are not orthogonal w.r.t. the standard inner product $\langle \cdot, \cdot \rangle_{\text{euclid}}$ since

$$\left\langle \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = 1 \cdot \overline{0} + \mathrm{i} \cdot \overline{1} = \mathrm{i} \neq 0.$$

However, there are orthogonal w.r.t. the inner product given by $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ with $A = \begin{pmatrix} 2 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}$, since

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\text{euclid}} = 0.$$



$x|_u$

The orthogonal projection of $\mathbf{x}$ onto $\mathrm{Span}(\mathbf{y})$ can be different for different inner products. W.r.t. $\langle \cdot, \cdot \rangle$ it is $\mathbf{o}$ (since $\mathbf{x} \perp \mathbf{y}$), but w.r.t. $\langle \cdot, \cdot \rangle_{\text{euclid}}$ it is

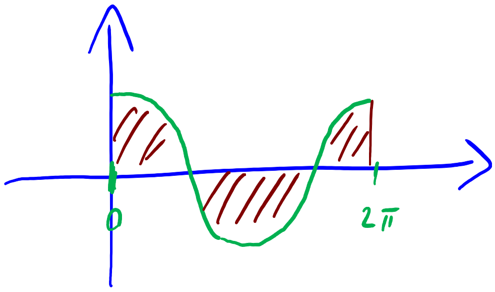
$$\mathbf{x}_{|\mathrm{Span}(\mathbf{y})} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}}{\langle \mathbf{y}, \mathbf{y} \rangle_{\text{euclid}}} \mathbf{y} = \frac{\langle \binom{1}{\mathrm{i}}, \binom{0}{1} \rangle_{\text{euclid}}}{\langle \binom{0}{1}, \binom{0}{1} \rangle_{\text{euclid}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\mathrm{i}}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{i} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix}.$$

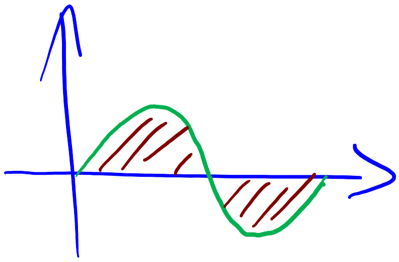
(b) Looking at the vector space $\mathcal{F}([0, 2\pi])$, which contains function $\mathbf{f} : [0, 2\pi] \to \mathbb{R}$, we define a subspace V that is spanned by the family $\mathcal{B} = (1, \cos, \sin)$. Then w.r.t the inner product defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle := \int_0^1 \mathbf{f}(x)\mathbf{g}(x)\, dx\,,$$

the family $\mathcal{B}$ is an OS:

$$\langle 1, \cos \rangle = \int_0^{2\pi} 1 \cdot \cos x \, dx = 0, \qquad \langle 1, \sin \rangle = \int_0^{2\pi} 1 \cdot \sin x \, dx = 0, \qquad \text{and}$$





$$\langle \cos, \sin \rangle = \int_0^{2\pi} (\cos x)(\sin x) dx = \tfrac{1}{2} \sin^2 x \Big|_0^{2\pi} = \frac{\sin^2 2\pi - \sin^2 0}{2} = 0.$$

Because of

mathematical analysis

$$\langle 1, 1 \rangle = \int_0^{2\pi} 1 \, dx = 2\pi, \quad \langle \cos, \cos \rangle = \int_0^{2\pi} \cos^2 x \, dx = \pi, \quad \langle \sin, \sin \rangle = \int_0^{2\pi} \sin^2 x \, dx = \pi$$

normalise

the new family $\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos}{\sqrt{\pi}}, \frac{\sin}{\sqrt{\pi}} \right)$ is an ONB of V.

$\mathcal{B} =$

↳ Remember audio signal

↳ With ONB, you can calculate $\Phi_B(f)$ easily

$f = \alpha_1 \cdot b_1 + \dots + \alpha_n \cdot b_n \quad \Rightarrow \quad \alpha_i = \langle f, b_i \rangle$

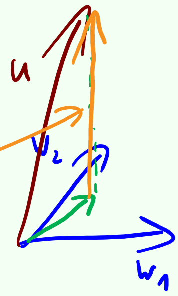
$\Phi_{\mathcal{B}}(f) = \begin{pmatrix} \langle f, b_1 \rangle \\ \vdots \\ \langle f, b_n \rangle \end{pmatrix}$

Remark: Gram-Schmidt orthonormalisation

Given: *Let V be a pre-Hilbert space and $\mathcal{C}$ a family of vectors from V.*
To Find: *An ONB $\mathcal{B}$ of* Span($\mathcal{C}$).

Algorithm:

```
Initialise B as the empty set ( );
For all u in C:
    Set v := u − u|Span(B);   ← normal component
    If v ≠ o:
        Set w := v/‖v‖;
        Add w to B
```



If you cancel the algorithm at some point, the family at this point, $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$, is a ONB of the Span$(\mathbf{w}_1, \ldots, \mathbf{w}_k)$.

Recall that for this ONB $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$ the orthogonal projection $\mathbf{u}_{|\mathrm{Span}(\mathcal{B})}$ is calculated by

$$\mathbf{u}_{|\mathrm{Span}(\mathcal{B})} = \langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \ldots + \langle \mathbf{u}, \mathbf{w}_k \rangle \mathbf{w}_k .$$

$f : [-1, 1] \to \mathbb{R}$
e.g. $f(x) = x^2 + x$

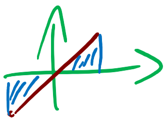
Example 7.41. The monomials $\mathcal{C} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ do not form an ONB in $\mathcal{P}([-1, 1])$ w.r.t. $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(x)\mathbf{g}(x)\, dx$. We can apply the Gram-Schmidt procedure for $\mathcal{C}$. Here it is useful to start with the numbering indices 0, 1, 2, ...

$$\mathbf{v}_0 = \mathbf{m}_0 = 1, \qquad \Longrightarrow \quad \mathbf{w}_0(x) = \frac{\mathbf{v}_0(x)}{\|\mathbf{v}_0\|} = \frac{1}{\sqrt{2}},$$

$\sqrt{\int_{-1}^{1} 1 \cdot dx} = \sqrt{2}$

$$\mathbf{v}_1 = \mathbf{m}_1 - \underbrace{\langle \mathbf{m}_1, \mathbf{w}_0 \rangle}_{0} \mathbf{w}_0 = \mathbf{m}_1, \qquad \Longrightarrow \quad \mathbf{w}_1(x) = \frac{\mathbf{v}_1(x)}{\|\mathbf{v}_1\|} = \sqrt{\frac{3}{2}}\, x,$$

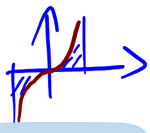
$0 = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx$



$\sqrt{\int_{-1}^{1} x^2\, dx} = \sqrt{\frac{2}{3}}$

$$\mathbf{v}_2 = \mathbf{m}_2 - \underbrace{\langle \mathbf{m}_2, \mathbf{w}_0 \rangle}_{\frac{\sqrt{2}}{3}} \mathbf{w}_0 - \underbrace{\langle \mathbf{m}_2, \mathbf{w}_1 \rangle}_{0} \mathbf{w}_1, \qquad \Longrightarrow \quad \mathbf{w}_2(x) = \frac{\mathbf{v}_2(x)}{\|\mathbf{v}_2\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).$$

$\frac{\sqrt{2}}{3} = \int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}}\, dx$ $\qquad 0 = c \cdot \int_{-1}^{1} x^3\, dx$



$\mathcal{B} = (\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ is an ONB for Span$(\mathcal{C}) = \mathcal{P}_2([-1, 1])$. The polynomials $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ (or also with other normalisation factors) are called the *Legendre polynomials*. If we add the other monomials $\mathbf{m}_3, \mathbf{m}_4, \ldots$, we get the next Legendre polynomials.

Summary

- *Vectors* are elements in a set, called a *vector space* V, that one can add together and scale with numbers α from $\mathbb{R}$ or $\mathbb{C}$, without leaving the set V. The addition and scalar multiplication just have to satisfy the rules (1)–(8) from Definition 7.1.

- If you know that a set V with two operations $+$ and $\alpha\cdot$ is a vector space and if you want to show that also a subset $U \neq \varnothing$ of V form a vector space, then you do not have to check (1)–(8) again, but only (a) and (b) from Proposition 7.7. This is called a *subspace* of V.
- The definitions *linear combination*, *span*, *generating system*, *linearly (in)dependent*, *basis* and *dimension* are literally the same in Chapter 3.
- If you fix a basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ in V, then each $\mathbf{x} \in V$ has a uniquely determined linear combination $\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$. The numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$) are called the *coordinates* of $\mathbf{x}$ w.r.t. $\mathcal{B}$. This defines the vector $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n$.
- Changing the basis of V from $\mathcal{B}$ to $\mathcal{C}$ also changes the coordinate vector from $\Phi_{cB}(\mathbf{x}) \in \mathbb{F}^n$ to $\Phi_{\mathcal{C}}(\mathbf{x}) \in \mathbb{F}^n$. This change can be describes by the *transformation matrix* $T_{\mathcal{C} \leftarrow \mathcal{B}}$.
- One always has $T_{\mathcal{B} \leftarrow \mathcal{C}} = T_{\mathcal{C} \leftarrow \mathcal{B}}^{-1}$. Sometimes, it is helpful to go a detour $T_{\mathcal{B} \leftarrow \mathcal{C}} = T_{\mathcal{B} \leftarrow \mathcal{A}} T_{\mathcal{A} \leftarrow \mathcal{C}}$ where $\mathcal{A}$ is a simple and well-known basis.
- An *inner product* $\langle \cdot, \cdot \rangle$ is a map, which takes two vectors $\mathbf{x}, \mathbf{y} \in V$ and gives out a number $\langle \mathbf{x}, \mathbf{y} \rangle$ in $\mathbb{F}$. It has to satisfy the rules (S1)–(S4) from Definition 7.23.
- If $A \in \mathbb{F}^{n \times n}$ is selfadjoint and *positive definite*, then $\langle \mathbf{x}, \mathbf{y} \rangle := \langle A\mathbf{x}, \mathbf{y} \rangle_{\text{euclid}}$ defines an inner product in $\mathbb{F}^n$. Here $\langle \cdot, \cdot \rangle_{\text{euclid}}$ is the well-known standard inner product in $\mathbb{R}^n$ (Chapter 2) or $\mathbb{C}^n$ (Chapter 6).
- A *norm* $\|\cdot\|$ is a map that sends a vector $\mathbf{x} \in V$ to number $\|\mathbf{x}\| \in \mathbb{R}$ and satisfy the rules (N1)–(N3) from Definition 7.33.
- An inner product $\langle \cdot, \cdot \rangle$ always defines a norm $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.
- By having an inner product, we can talk about orthogonal projection $\mathbf{x}_{|U}$ for a vector $\mathbf{x} \in V$ w.r.t. a subspace $U \subset V$.

8 General linear maps

V W

$f: \mathbb{R}^n \to \mathbb{R}^m$

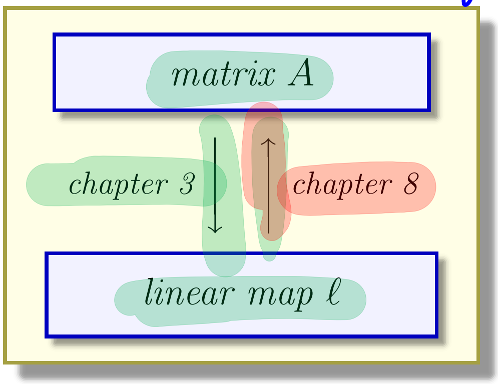
conserves the addition and scalar multiplication

We have also seen that for a given matrix $A \in \mathbb{R}^{m \times n}$ there is an associated map

$$f_A : \mathbb{R}^n \to \mathbb{R}^m \qquad \text{with} \qquad \mathbf{x} \mapsto A\mathbf{x}$$

which fulfils two properties $(+)$ and $(\cdot)$ and therefore is called a linear map.

$f_A(x+y) = f_A(x) + f_A(y)$
$f_A(\lambda \cdot x) = \lambda f_A(x)$



Now in Chapter 8, with the power of general vector spaces, we also can consider general linear maps between arbitrary $\mathbb{F}$-vector spaces V and W.

how to store this information in a matrix?

8.1 Definition: Linear maps

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ again. Let V and W be two $\mathbb{F}$-vector spaces. It is important that for both the same field $\mathbb{F}$ is chosen.

Definition 8.1. Linear map

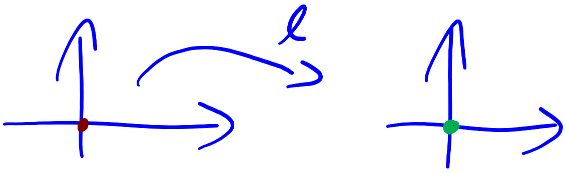
A map $\ell : V \to W$ is called a linear map, linear function or linear operator if ℓ satisfies the two following properties. For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$:

+ in V   + in W

$$(L+) \qquad \ell(\mathbf{x} + \mathbf{y}) = \ell(\mathbf{x}) + \ell(\mathbf{y}), \qquad \textit{(additive)}$$
$$(L\,\cdot) \qquad \ell(\alpha\mathbf{x}) = \alpha\,\ell(\mathbf{x}). \qquad \textit{(homogeneous)}$$

in V   in W

If $W = \mathbb{F}$, one often calls ℓ a linear functional.



Proposition 8.2. Linear maps send o to o.

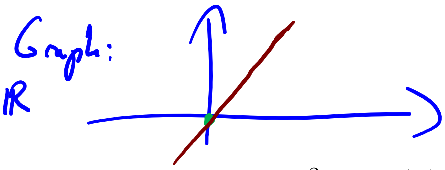
For a linear map $\ell : V \to W$, we have $\ell(\mathbf{o}_V) = \mathbf{o}_W$.

Proof. For arbitrary $\mathbf{x} \in V$, we use (L·): $\ell(\mathbf{o}_V) = \ell(0\mathbf{x}) = 0\,\ell(\mathbf{x}) = \mathbf{o}_W$. $\square$

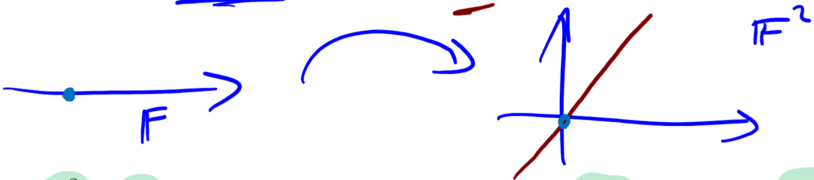
In the following examples $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$.

Example 8.3. (a) For $V = W = \mathbb{F}$, let $\ell(\mathbf{x}) = 3\mathbf{x}$. We can easily check (L+) and (L·).

$\ell : \mathbb{R} \to \mathbb{R}$



(b) For $V = \mathbb{F}$ and $W = \mathbb{F}^2$, let $\ell(x) = x\binom{3}{1}$. Obviously, ℓ satisfies (L+) and (L·).



(c) Let $\ell : \mathbb{F}^3 \to \mathbb{F}$ defined by $\ell(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle_{\text{euclid}} = \mathbf{a}^*\mathbf{x}$ with fixed $\mathbf{a} \in \mathbb{F}^3$, e.g.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \text{hence} \qquad \ell : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\rangle_{\text{euclid}} = \underbrace{(2\ 1\ 3)}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

$\ell = f_A$

Using the definition of an inner product, we know that ℓ is linear.

(d) Define $\ell : \mathbb{F}^3 \to \mathbb{F}$ by $\ell(\mathbf{x}) = \det(\mathbf{x}\ \mathbf{a}_2\ \mathbf{a}_3)$ with fixed $\mathbf{a}_2, \mathbf{a}_3 \in \mathbb{F}^3$, e.g.

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{hence} \qquad \ell : \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \det \begin{pmatrix} | & | & | \\ \mathbf{x} & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{pmatrix} = \det \begin{pmatrix} x_1 & 1 & 3 \\ x_2 & 0 & 1 \\ x_3 & 2 & 1 \end{pmatrix}.$$

We know from the definition of the determinant that ℓ is linear. Using Laplace's formula, we can rewrite ℓ:

$$\ell(\mathbf{x}) = x_1 \underbrace{\det \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}}_{-2} - x_2 \underbrace{\det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}}_{-5} + x_3 \underbrace{\det \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}_{1} = \underbrace{(-2\ 5\ 1)}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\ell = f_A$

(e) The map $\ell : \mathbb{F}^2 \to \mathbb{F}^2$ defined by $\binom{x_1}{x_2} \mapsto \binom{4x_1+3x_2}{x_2+7}$ is not linear because $\ell(\mathbf{o}) = \binom{0}{7} \neq \mathbf{o}$.

(f) For $A \in \mathbb{F}^{m\times n}$ define $f_A : \mathbb{F}^n \to \mathbb{F}^m$ by $f_A : \mathbf{x} \mapsto A\mathbf{x}$. This is a linear map by Proposition 3.14. For example, $\mathbb{F} = \mathbb{R}$ and $m = n = 2$, look at how f_A acts on houses.

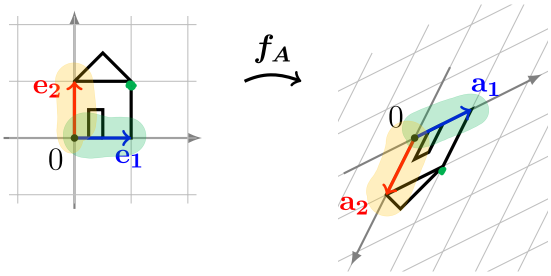
Let

$$A = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

We know:

$$\mathbf{o} \overset{f_A}{\mapsto} \mathbf{o}, \quad \mathbf{e}_1 \overset{f_A}{\mapsto} \mathbf{a}_1, \quad \mathbf{e}_2 \overset{f_A}{\mapsto} \mathbf{a}_2$$

and the rest of the plane is given by linearity.



The last example (f) includes all the other examples (a)–(e): We always find a corresponding matrix $\ell(\mathbf{x}) = A\mathbf{x}$.

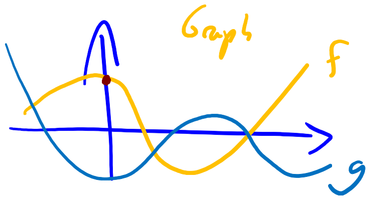
$$A = \begin{pmatrix} | & & | \\ \ell(e_1) & \cdots & \ell(e_n) \\ | & & | \end{pmatrix} \quad \text{for } \ell : \mathbb{F}^n \to \mathbb{F}^m$$

$$\ell = f_A$$

How does this work for $\ell : V \to W$?

Now let us look for some abstract vector spaces:

Example 8.4. (a) Let $V = \mathcal{F}(\mathbb{R})$, $W = \mathbb{R}$ and $\delta_0 : V \to W$ the evaluation for a function $\mathbf{f} \in V$ in the origin 0, which means $\delta_0 : \mathbf{f} \mapsto \mathbf{f}(0)$. Then δ_0 is linear. (Show it!) ✓



$$\delta_0(f) = f(0) \qquad \delta_0(f+g) = (f+g)(0)$$
$$= f(0) + g(0) = \delta_0(f) + \delta_0(g) \checkmark$$

Another example would be a evaluation at different points and using linear combinations: $\ell : \mathbf{f} \mapsto 3\mathbf{f}(0) - 7\mathbf{f}(\frac{1}{4}) + 5\mathbf{f}(1)$.

(b) Let ∂ be the differential operator from $V = \mathcal{P}_3(\mathbb{R})$ to $W = \mathcal{P}_2(\mathbb{R})$, which means ∂ sends a polynomial $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ to its derivative $\mathbf{f}' \in \mathcal{P}_2(\mathbb{R})$.

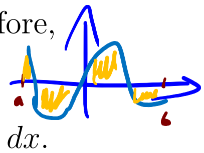
$$\partial : m_2 \mapsto 2m_1$$
$$f(x) = x^2, \quad f'(x) = 2x$$
$$(f+g)' = f' + g' \quad (\text{from school})$$

Second derivative

(c) In the same manner, we can look at the map $\mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ with $\mathbf{f} \mapsto \mathbf{f}''$ given by the second derivative. In the same way, a combination is possible, $\mathbf{f} \mapsto \mathbf{f}''' + 3\mathbf{f}'' - 2\mathbf{f}' + 4\mathbf{f}$

as a map $\mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$.

(d) Instead of using the derivative of a polynomial $\mathbf{f} \in \mathcal{P}([a,b]) =: V$ or evaluating it in one point, we can use the integration, hence the map $i : \mathbf{f} \mapsto \int_a^b \mathbf{f}(x)\,dx$. Therefore, in this case, we have $V = \mathcal{P}([a,b])$ and $W = \mathbb{R}$. Again, we get a linear map:



$$\int_a^b \big(\mathbf{f}(x) + \mathbf{g}(x)\big)\,dx = \int_a^b \mathbf{f}(x)\,dx + \int_a^b \mathbf{g}(x)\,dx \quad \text{and} \quad \int_a^b \alpha\mathbf{f}(x)\,dx = \alpha \int_a^b \mathbf{f}(x)\,dx.$$

(We also talk about the integration in mathematical analysis next semester.)

→ For δ_0, ∂, i, we will find also matrices to store the information!

$\boxed{\text{Need: } \Phi_B : V \to \mathbb{F}^n}$

8.2 Combinations of linear maps

8.2.1 Sum and multiples of a linear map

Definition 8.5. Sum and scaled linear maps

Let V and W be two $\mathbb{F}$-vector space (with same $\mathbb{F}$!) and let $k : V \to W$ and $\ell : V \to W$ be linear maps. Then we define $k+\ell : V \to W$ by

$$(k+\ell)(\mathbf{x}) := k(\mathbf{x}) + \ell(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \in V,$$

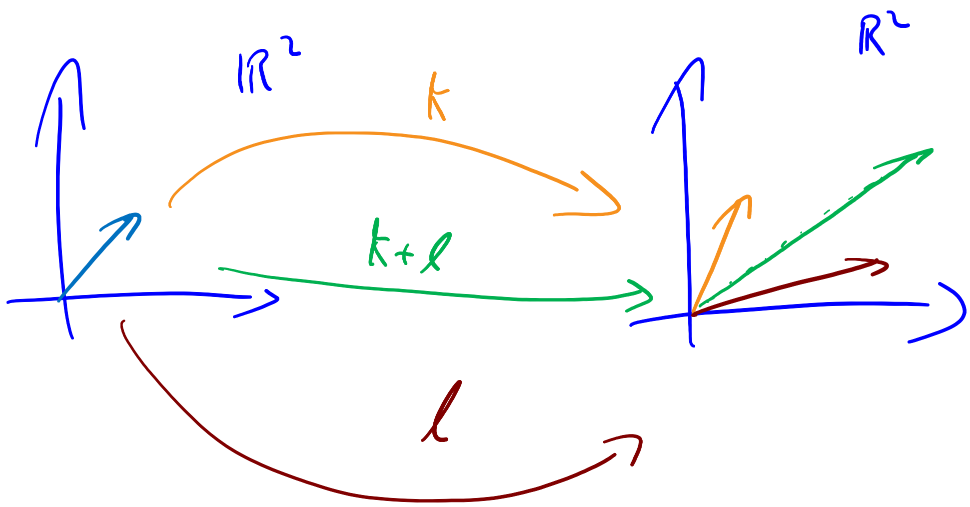
+ in W

and for $\alpha \in \mathbb{F}$, we define $\alpha \cdot \ell$ by

· in W

$$(\alpha \cdot \ell)(\mathbf{x}) := \alpha \cdot \ell(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \in V.$$

The operations $+$ and $\alpha\cdot$ on the right-hand side are the operations in W.



Proposition & Definition 8.6. Vector space of linear maps $V \to W$

The maps $k+\ell$ and $\alpha \cdot \ell$ from Definition 8.5 are again linear maps from V to W.

The set of all linear maps from V to W equipped with the two operations $+$ and $\alpha \cdot$ form again an $\mathbb{F}$-vector space. We denote this vector space by $\mathcal{L}(V,W)$.

The zero vector in $\mathcal{L}(V,W)$ is the zero map $o : V \to W$ defined by $o(\mathbf{x}) = \mathbf{o}$ for all $\mathbf{x} \in V$.

Proof. Let $k, \ell : V \to W$ be linear and let $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$. Then:

$$\begin{aligned} (k+\ell)(\mathbf{x}+\mathbf{y}) &\overset{Def. 8.5}{=} k(\mathbf{x}+\mathbf{y}) + \ell(\mathbf{x}+\mathbf{y}) \overset{(L+)}{=} k(\mathbf{x}) + k(\mathbf{y}) + \ell(\mathbf{x}) + \ell(\mathbf{y}) \\ &= k(\mathbf{x}) + \ell(\mathbf{x}) + k(\mathbf{y}) + \ell(\mathbf{y}) \overset{Def. 8.5}{=} (k+\ell)(\mathbf{x}) + (k+\ell)(\mathbf{y}) \end{aligned}$$

$$\text{and} \qquad \begin{aligned} (k+\ell)(\alpha\mathbf{x}) &\overset{Def. 8.5}{=} k(\alpha\mathbf{x}) + \ell(\alpha\mathbf{x}) \overset{(L\cdot)}{=} \alpha\, k(\mathbf{x}) + \alpha\, \ell(\mathbf{x}) = \alpha\big(k(\mathbf{x}) + \ell(\mathbf{x})\big) \\ &\overset{Def. 8.5}{=} \alpha(k+\ell)(\mathbf{x}), \end{aligned}$$

which means $k+\ell$ has two properties (L+) and (L·) and is also linear. In the same manner, we see that $\alpha \cdot \ell$ is linear. Show (1)-(8) as an exercise! □

From now on, we do not write the two operations $+$ and $\alpha \cdot$ in $\mathcal{L}(V,W)$ in red anymore. However, keep in mind that these are different operations than $+$ and $\alpha \cdot$ in W.

Example 8.7. – Projection and reflection. Let $\mathbf{n} \in \mathbb{R}^n$ be a vector $\|\mathbf{n}\| = 1$ and $G := \mathrm{Span}(\mathbf{n})$ the spanned line. For all $\mathbf{x} \in \mathbb{R}^n$, we can calculate the orthogonal projection

$$\mathbf{x}_{|G} = \frac{\langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}}}{\langle \mathbf{n}, \mathbf{n} \rangle_{\text{euclid}}} \mathbf{n} = \langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}} \mathbf{n} = \mathbf{n} \langle \mathbf{x}, \mathbf{n} \rangle_{\text{euclid}} = \mathbf{n}(\mathbf{n}^\top \mathbf{x}) = (\mathbf{n}\mathbf{n}^\top)\mathbf{x}$$

A Matrix multiplication

.

Hence the map

$$\mathrm{proj}_G : \mathbb{R}^n \to \mathbb{R}^n \qquad \textit{with} \qquad \mathrm{proj}_G(\mathbf{x}) := \mathbf{x}_{|G} = (\mathbf{n}\mathbf{n}^\top)\mathbf{x}, \tag{8.1}$$

defines a linear map $\mathbb{R}^n \to \mathbb{R}^n$. We also know that is given by the associated matrix: $\mathrm{proj}_G = f_{\mathbf{n}\mathbf{n}^\top}$.

Using the orthogonal decomposition

$$\mathbf{x} \;=\; \mathbf{x}_{|G} \;+\; \mathbf{x}_{|E}\,,$$

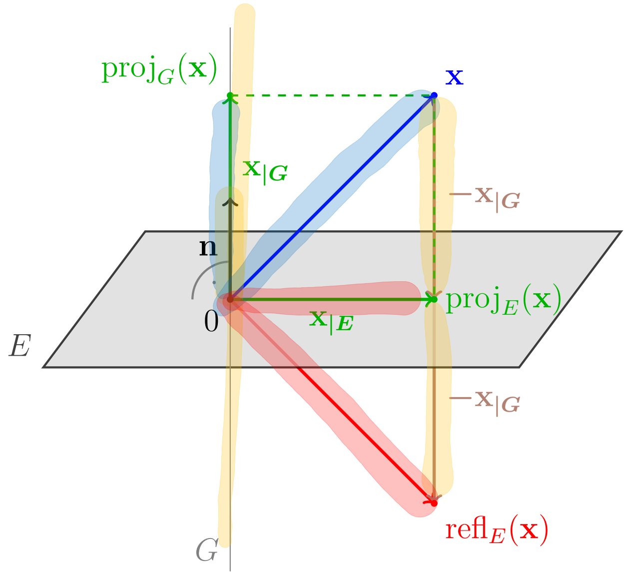
we also can also define the linear map

$$\mathrm{proj}_E : \mathbb{R}^n \to \mathbb{R}^n$$

which is the orthogonal projection onto $E := G^\perp = \{\mathbf{n}\}^\perp$:

$$\boxed{\mathrm{proj}_E(\mathbf{x}) := \mathbf{x}_{|E} = \mathbf{x} - \mathbf{x}_{|G}.}$$

Subtracting the orthogonal projection $\mathbf{x}_{|G}$ again, we get the reflection of $\mathbf{x}$ with respect to the hyperplane E.



Hence, we define:

$$\boxed{\mathrm{refl}_E : \mathbb{R}^n \to \mathbb{R}^n \qquad \textit{with} \qquad \mathrm{refl}_E(\mathbf{x}) := \mathbf{x}_{|E} - \mathbf{x}_{|G} = \mathbf{x} - 2\mathbf{x}_{|G}.}$$

In other words:

$$\mathrm{proj}_E = id - \mathrm{proj}_G \qquad \text{and} \qquad \mathrm{refl}_E = id - 2\,\mathrm{proj}_G. \tag{8.2}$$

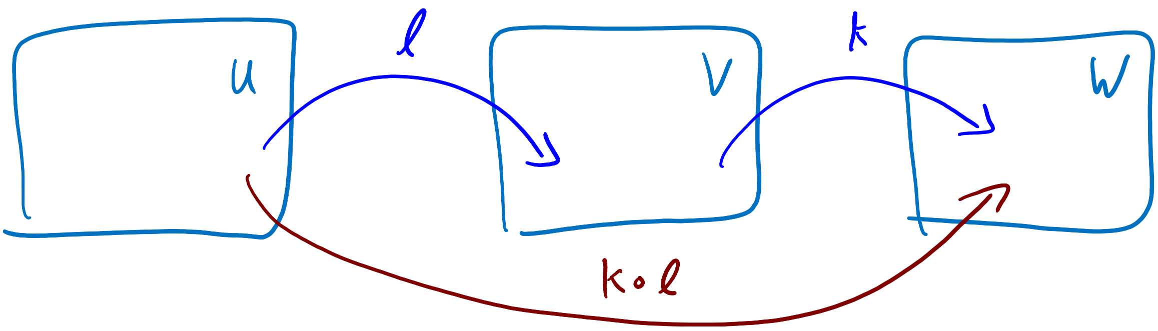
Here, $id : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map $id : \mathbf{x} \mapsto \mathbf{x}$. By these formulas, we can conclude, $\mathrm{proj}_E, \mathrm{refl}_E \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

$(id = f_{\mathbb{1}})$

8.2.2 Composition and inverses

Recall that you can form the composition of two maps $\ell : U \to V$ and $k : V \to W$ by setting:

$$(k \circ \ell)(\mathbf{x}) \;=\; k(\ell(\mathbf{x})) \qquad \text{for all} \quad \mathbf{x} \in U. \tag{8.3}$$



Proposition 8.8. Composition of linear maps is linear.

Let U, V, W be $\mathbb{F}$-vector spaces and let $\ell : U \to V$ and $k : V \to W$ be linear maps. Then, the composition $k \circ \ell : U \to W$ is also linear. In short:

$$\ell \in \mathcal{L}(U,V),\;\; k \in \mathcal{L}(V,W) \;\;\Rightarrow\;\; k \circ \ell \in \mathcal{L}(U,W)\,.$$

Let compose the maps from Example 8.7:

Important: $k \circ (\ell_1 + \ell_2) = (k \circ \ell_1) + (k \circ \ell_2)$ Exercise!

$(k_1 + k_2) \circ \ell = (k_1 \circ \ell) + (k_2 \circ \ell)$

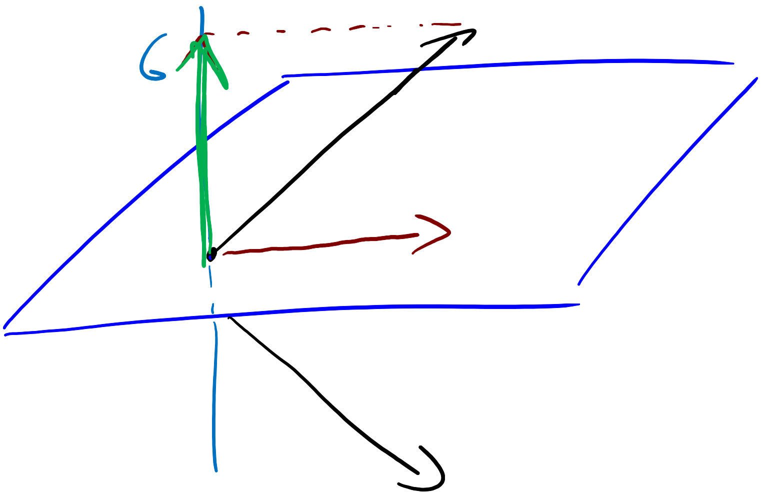
Example 8.9. Recall both projections

$$\mathrm{proj}_G : \mathbf{x} \mapsto \mathbf{n}\mathbf{n}^\top \mathbf{x} \qquad \text{and} \qquad \mathrm{proj}_E = id - \mathrm{proj}_G$$

and the reflection

$$\mathrm{refl}_E = id - 2\,\mathrm{proj}_G.$$

All three maps act between $\mathbb{R}^n \to \mathbb{R}^n$ and can be composed in all possible ways.



$\mathrm{proj}_G(\mathrm{proj}_G(x)) = \mathrm{proj}_G(x)$

$\mathrm{proj}_E(\mathrm{proj}_G(x)) = 0$

We already know that projecting more than once does not change anything:

$$\mathrm{proj}_G \circ \mathrm{proj}_G = \mathrm{proj}_G \qquad \text{and} \qquad \mathrm{proj}_E \circ \mathrm{proj}_E = \mathrm{proj}_E. \tag{8.4}$$

For the reflection, we expect that using it two times brings us back to the beginning, which means that we should get the identity map:

$$\begin{aligned} \mathrm{refl}_E \circ \mathrm{refl}_E &= (id - 2\,\mathrm{proj}_G) \circ (id - 2\,\mathrm{proj}_G) \\ &= \underbrace{id \circ id}_{id} - \underbrace{id \circ 2\,\mathrm{proj}_G}_{2\,\mathrm{proj}_G} - \underbrace{2\,\mathrm{proj}_G \circ id}_{2\,\mathrm{proj}_G} + \underbrace{2\,\mathrm{proj}_G \circ 2\,\mathrm{proj}_G}_{4\,\mathrm{proj}_G} = id. \end{aligned}$$

Composition of both projections gives us the zero map:

$$\mathrm{proj}_G \circ \mathrm{proj}_E = \mathrm{proj}_G \circ (id - \mathrm{proj}_G) = \underbrace{\mathrm{proj}_G \circ id}_{\mathrm{proj}_G} - \underbrace{\mathrm{proj}_G \circ \mathrm{proj}_G}_{\mathrm{proj}_G} = o. \tag{8.5}$$

In the same way, $\mathrm{proj}_E \circ \mathrm{proj}_G = o$. We also can calculate:

$$\mathrm{refl}_E \circ \mathrm{proj}_G = -\mathrm{proj}_G \qquad \text{and} \qquad \mathrm{refl}_E \circ \mathrm{proj}_E = \mathrm{proj}_E. \tag{8.6}$$

Changing the order gives us the same result.

We again look at more abstract examples:

Example 8.10. (a) Let $\delta_0 : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ given by $\delta_0 : \mathbf{f} \mapsto \mathbf{f}(0)$ the point evaluation and $\partial : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ the differential operator $\partial : \mathbf{f} \mapsto \mathbf{f}'$ from Example 8.4 (a) and (b). Then, the composition $\delta_0 \circ \partial$ from $\mathcal{P}_3(\mathbb{R})$ to $\mathbb{R}$ is given by

$\mathcal{P}_3(\mathbb{R}) \xrightarrow{\partial} \mathcal{P}_2(\mathbb{R}) \xrightarrow{\delta_0} \mathbb{R}$

$$\mathbf{f} \overset{\partial}{\mapsto} \mathbf{f}' \overset{\delta_0}{\mapsto} \mathbf{f}'(0), \qquad \text{hence} \qquad \delta_0 \circ \partial : \mathbf{f} \mapsto \mathbf{f}'(0).$$

The reverse composition $\partial \circ \delta_0$ is not defined!

(b) Let $\partial : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the differentiation $\mathbf{f} \mapsto \mathbf{f}'$ and, in addition, $\int : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ the map that sends $\mathbf{f} \in \mathcal{P}_2(\mathbb{R})$ to the function $\mathbf{F}$ with

$$\mathbf{F}(x) = \int_0^x \mathbf{f}(t)\,dt \qquad \text{for all} \quad x \in [0,1]\,.$$

We get:

$$\mathbf{f} \overset{\int}{\mapsto} \mathbf{F} \overset{\partial}{\mapsto} \mathbf{F}' = \mathbf{f}, \quad \text{hence} \quad \partial \circ \textstyle\int : \mathbf{f} \mapsto \mathbf{f}, \quad \text{which means} \quad \partial \circ \textstyle\int = id : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}).$$

We can also build the converse composition of ∂ and $\int$. Is $\int \circ \partial$ then the identity map $id : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$?

Let $\mathbf{f} \in \mathcal{P}_3(\mathbb{R})$ be arbitrary, which means $\mathbf{f}(x) = ax^3 + bx^2 + cx + d$ with some $a, b, c, d \in \mathbb{R}$. Then $\partial(\mathbf{f}) = \mathbf{f}'$ with $\mathbf{f}'(x) = 3ax^2 + 2bx + c$. Now, we use $\int$: The function $\mathbf{g} := (\int \circ \partial)(\mathbf{f}) = \int(\partial(\mathbf{f})) = \int(\mathbf{f}')$ satisfies:

$$\mathbf{g}(x) = \int_0^x \mathbf{f}'(t)\,dt = \int_0^x \big(3at^2 + 2bt + c\big)\,dt = at^3 + bt^2 + ct\Big|_0^x = ax^3 + bx^2 + cx$$

for all x. Hence, $(\int \circ \partial)(\mathbf{f}) \neq \mathbf{f}$ if $d \neq 0$. We see that "$+d$" is lost. We conclude $\int \circ \partial \neq id$.

Reminder: Inverse maps

We call a map $f : V \to W$ invertible if there is another map $g : W \to V$ with

$$f \circ g = id_W \qquad \text{and} \qquad g \circ f = id_V$$

Since g uniquely determined, it is called the inverse map of f and denoted by f^{-1}.

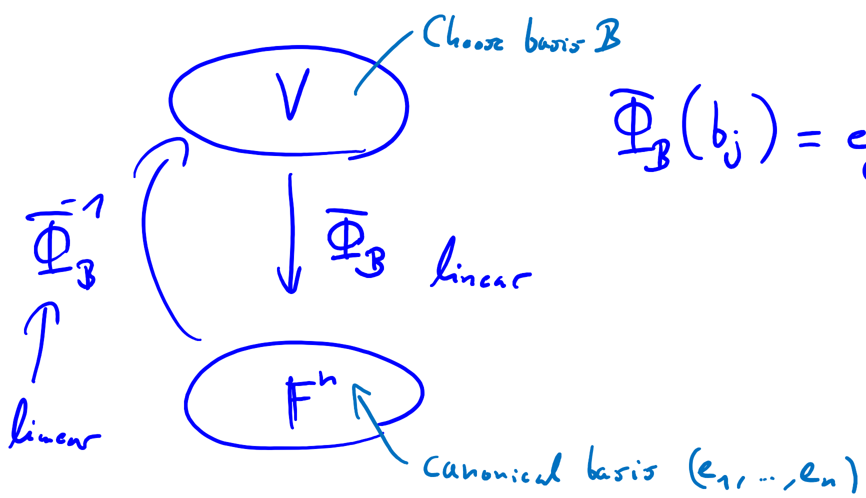
Recall that *bijective* and *invertible* are equivalent notions for maps.

However, here, we are only interested in linear maps between vector spaces. As mentioned in Chapter 3, we have the following interesting result:

Proposition 8.11. Inverses are again linear.

If $\ell : V \to W$ is a linear map that is bijective, then its inverse $\ell^{-1} : W \to V$ is also linear

Example 8.12. Recall that we already considered a linear map in Section 7.4, namely the map $\Phi_{\mathcal{B}} : \mathbf{v} \mapsto \mathbf{v}^{\mathcal{B}}$, which maps a vector $\mathbf{v}$ from an $\mathbb{F}$-vector space V to its coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ with respect to a basis $\mathcal{B}$.



$\Phi_{\mathcal{B}}(b_j) = e_j$

surjective?

injective?

$\Uparrow$ (Maple)

lin. indp. $\rightsquigarrow$ lin. indep.

Remark:

A linear map $\ell : V \to W$ exactly conserves the structure of the vector spaces, meaning vector addition and scalar multiplication. Therefore, mathematicians call a linear map a homomorphism. A homomorphism ℓ that is invertible and has an inverse ℓ^{-1} that is also a homomorphism is called an isomorphism.

8.3 Finding the matrix for a linear map

8.3.1 Just know what happens to a basis

Rule of thumb: Linearity makes it easy

For a linear map, you only have to know what happens to a basis. The remaining part of space "tags along".

Let $\ell : V \to W$ be a linear map and $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ some basis of V. For each $\mathbf{x} \in V$, we denote by $\Phi_{\mathcal{B}}(\mathbf{x}) \in \mathbb{F}^n$ its coordinate vector, which means

$$\Phi_{\mathcal{B}}(\mathbf{x}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n \quad \text{with} \quad \mathbf{x} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n = \Phi_{\mathcal{B}}^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} .$$

Then:

$$\boxed{\ell(\mathbf{x}) = \ell(\alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n) = \alpha_1 \ell(\mathbf{b}_1) + \dots + \alpha_n \ell(\mathbf{b}_n)}$$

Equation (8.7) says everything: If you know the images of the all basis elements, which means $\ell(\mathbf{b}_1), \dots, \ell(\mathbf{b}_n)$, then you know all images $\ell(\mathbf{x})$ for each $\mathbf{x} \in V$ immediately.

Example 8.13. Let $V = \mathcal{P}_3(\mathbb{R})$ with the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ where $\mathbf{m}_k(x) = x^k$. For the differential operator $\partial \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ where $\partial : \mathbf{f} \mapsto \mathbf{f}'$, we have

$$\partial(\mathbf{m}_0) = \mathbf{o}, \quad \partial(\mathbf{m}_1) = \mathbf{m}_0, \quad \partial(\mathbf{m}_2) = 2\mathbf{m}_1, \quad \partial(\mathbf{m}_3) = 3\mathbf{m}_2, \tag{8.7}$$

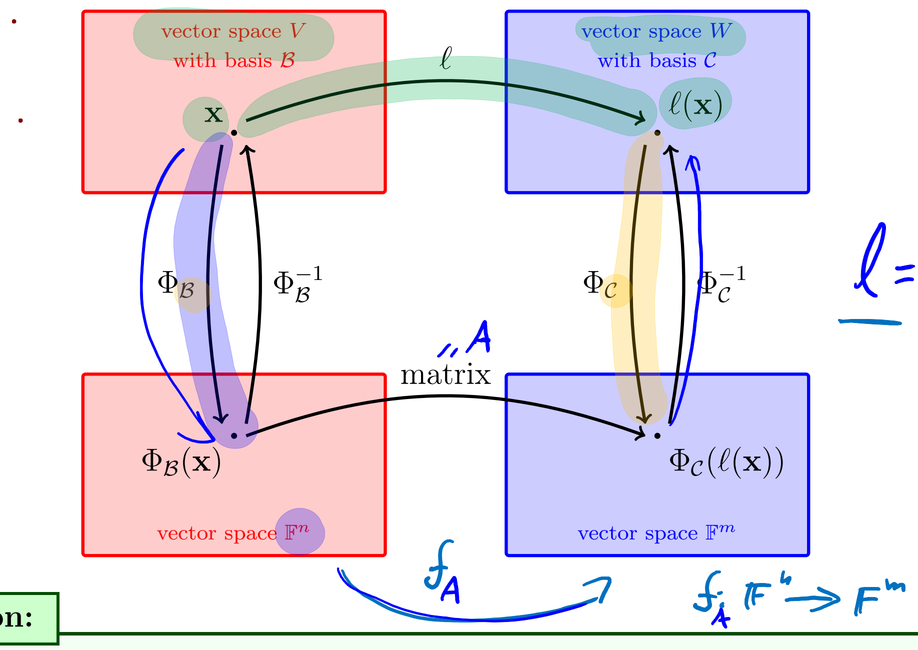
For an arbitrary $\mathbf{p} \in \mathcal{P}_3(\mathbb{R})$, which means $\mathbf{p}(x) = ax^3 + bx^2 + cx + d$ for $a, b, c, d \in \mathbb{R}$ or $\mathbf{p} = d\mathbf{m}_0 + c\mathbf{m}_1 + b\mathbf{m}_2 + a\mathbf{m}_3$, we have

$$\mathbf{p}^{\mathcal{B}} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \text{ and hence } \partial(\mathbf{p}) = d\partial(\mathbf{m}_0) + c\partial(\mathbf{m}_1) + b\partial(\mathbf{m}_2) + a\partial(\mathbf{m}_3) = c\mathbf{m}_0 + 2b\mathbf{m}_1 + 3a\mathbf{m}_2.$$

Checking this: $\mathbf{p}'(x) = 3ax^2 + 2bx + c$, hence $\partial(\mathbf{p}) = \mathbf{p}' = 3a\mathbf{m}_2 + 2b\mathbf{m}_1 + c\mathbf{m}_0$.

8.3.2 Matrix of a linear map with respect to bases

Let us consider again two arbitrary finite-dimensional $\mathbb{F}$-vector spaces V and W and linear maps between them.



$\ell = \Phi_{\mathcal{C}}^{-1} \circ f_A \circ \Phi_{\mathcal{B}}$

all informations of ℓ are in A

Question:

How to get the map or the matrix in the bottom. How to send the coordinate vector $\Phi_{\mathcal{B}}(\mathbf{x})$ to the coordinate vector $\Phi_{\mathcal{C}}(\ell(\mathbf{x}))$?

Of course, this is given by composing the three maps:

$$\Phi_{\mathcal{C}}(\ell(\mathbf{x})) = (\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1})(\Phi_{\mathcal{B}}(\mathbf{x}))$$

$$(\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1})(\mathbf{e}_j) = \Phi_{\mathcal{C}}(\ell(\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j))) = \Phi_{\mathcal{C}}(\ell(\mathbf{b}_j))$$

This gives us a matrix that really represents the abstract linear map. It depends, of course, on the chosen bases $\mathcal{B}$ and $\mathcal{C}$ in the vector spaces V and W, respectively. Therefore, we choose a good name:

Matrix representation of the linear map

For the linear map $\ell : V \to W$, we define the matrix

$$\ell_{\mathcal{C}\leftarrow\mathcal{B}} := \begin{pmatrix} | & & | \\ \Phi_{\mathcal{C}}(\ell(\mathbf{b}_1)) & \cdots & \Phi_{\mathcal{C}}(\ell(\mathbf{b}_n)) \\ | & & | \end{pmatrix} \in \mathbb{F}^{m\times n} \tag{8.8}$$

and call it the matrix representation of the linear map ℓ with respect to the basis $\mathcal{B}$ and $\mathcal{C}$.