Proposition & Definition 7.14. Monomial basis of $\mathcal{P}_n(\mathbb{R})$

Let $n \in \mathbb{N}_0$. The particular polynomials $\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n \in \mathcal{P}_n(\mathbb{R})$ defined by

$$\mathbf{m}_0(x) = 1, \quad \mathbf{m}_1(x) = x, \quad \ldots, \quad \mathbf{m}_{n-1}(x) = x^{n-1}, \quad \mathbf{m}_n(x) = x^n \quad \text{for all } x \in \mathbb{R}$$

are called monomials. The family $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n)$ forms a basis of $\mathcal{P}_n(\mathbb{R})$ and is called the monomial basis. Hence $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$.

Corollary 7.15. The method of equating the coefficients

Let $\mathbf{p}$ and $\mathbf{q}$ be two real polynomials with degree $n \in \mathbb{N}$, which means

$$\mathbf{p}(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad \text{and} \qquad \mathbf{q}(x) = b_n x^n + \ldots + b_1 x + b_0$$

for some coefficients $a_n, \ldots, a_1, a_0, b_n, \ldots, b_1, b_0 \in \mathbb{R}$.
If we have the equality $\mathbf{p} = \mathbf{q}$, which means

$$a_n x^n + \ldots + a_1 x + a_0 = b_n x^n + \ldots + b_1 x + b_0, \tag{7.3}$$

for all $x \in \mathbb{R}$, then we can conclude $a_n = b_n, \ \ldots, \ a_1 = b_1$ and $a_0 = b_0$.

A basis of

$\mathcal{P}_n(\mathbb{R})$: $(m_0, m_1, \ldots, m_n)$

$\dim(\mathcal{P}_n(\mathbb{R})) = n+1$

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Remark:

Since $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$ and we have the inclusions

$$\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R}) \subset \mathcal{P}_2(\mathbb{R}) \subset \cdots \subset \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}),$$

we conclude that $\dim(\mathcal{P}(\mathbb{R}))$ and $\dim(\mathcal{F}(\mathbb{R}))$ cannot be finite natural numbers. Symbolically, we write $\dim(\mathcal{P}(\mathbb{R})) = \infty$ in such a case.

We focuse on vector spaces with $\dim(V) < \infty$

7.4 Coordinates with respect to a basis

7.4.1 Basis implies coordinates

Again, we deal with the case $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ simultaneously. Therefore, let V be an $\mathbb{F}$-vector space with the two operations $+$ and $\cdot$. Let also $n := \dim(V) < \infty$ and choose a basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of V.

Since $\mathcal{B}$ is a generating system and linearly independent, each $\mathbf{v}$ from V has a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \ldots + \alpha_n \mathbf{b}_n \tag{7.4}$$

where the coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ are uniquely determined. We call these numbers the coordinates of $\mathbf{v}$ with respect to the basis $\mathcal{B}$ and sometimes write $\mathbf{v}^{\mathcal{B}}$ for the vector consisting of these numbers:

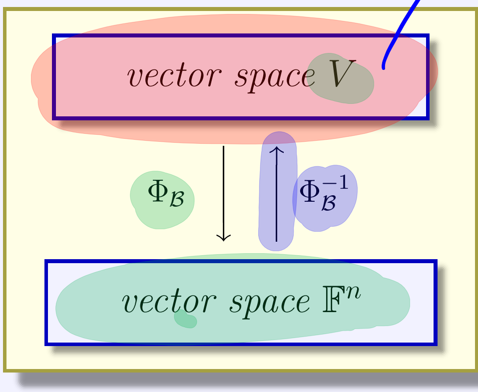
A vector $\mathbf{v}$ in V and its coordinate vector $\mathbf{v}^{\mathcal{B}}$ in $\mathbb{F}^n$

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \ldots + \alpha_n \mathbf{b}_n \in V \quad \longleftrightarrow \quad \mathbf{v}^{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n. \tag{7.5}$$

When fixing a basis $\mathcal{B}$ in V, then each vector $\mathbf{v} \in V$ uniquely determines a coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ – and vice versa.

Forming the coordinate vector is a linear map

Choose basis $\mathcal{B}$



The translation of a vector $\mathbf{v} \in V$ into the coordinate vector $\mathbf{v}^{\mathcal{B}} \in \mathbb{F}^n$ defines a linear map:

$$\Phi_{\mathcal{B}} : V \to \mathbb{F}^n, \quad \Phi_{\mathcal{B}}(\mathbf{v}) = \mathbf{v}^{\mathcal{B}}$$

standard basis of $\mathbb{F}^n$

More concretely:

$$\Phi_{\mathcal{B}}(\alpha_1 \mathbf{b}_1 + \ldots + \alpha_n \mathbf{b}_n) = \alpha_1 \mathbf{e}_1 + \ldots + \alpha_n \mathbf{e}_n$$

For all $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in \mathbb{F}$, the map Φ satisfies two properties:

$$\Phi_{\mathcal{B}}(\mathbf{x} + \mathbf{y}) = \Phi_{\mathcal{B}}(\mathbf{x}) + \Phi_{\mathcal{B}}(\mathbf{y}) \tag{+}$$

$$\Phi_{\mathcal{B}}(\lambda \mathbf{x}) = \lambda \Phi_{\mathcal{B}}(\mathbf{x}) \tag{$\cdot$}$$

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \ldots + \alpha_n \mathbf{b}_n \in V \quad \longleftrightarrow \quad \Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n. \tag{7.6}$$

The linear map $\Phi_{\mathcal{B}}$ is called the basis isomorphism with respect to the basis $\mathcal{B}$ and is completely defined by $\Phi(\mathbf{b}_j) = \mathbf{e}_j$ for $j = 1, \ldots, n$.

(and (+) and (·))

$\alpha \sin(x) + \beta \cos(x) + \gamma \cdot \arctan(x) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow \alpha = \beta = \gamma = 0 \Rightarrow$ lin. indep.

Example 7.16. An abstract vector is represented by numbers

The three functions sin, cos *and* arctan *from the vector space* $\mathcal{F}(\mathbb{R})$, *cf. Example 7.4, span a subspace:*

$$V := \mathrm{Span}(\sin, \cos, \arctan) \subset \mathcal{F}(\mathbb{R}).$$

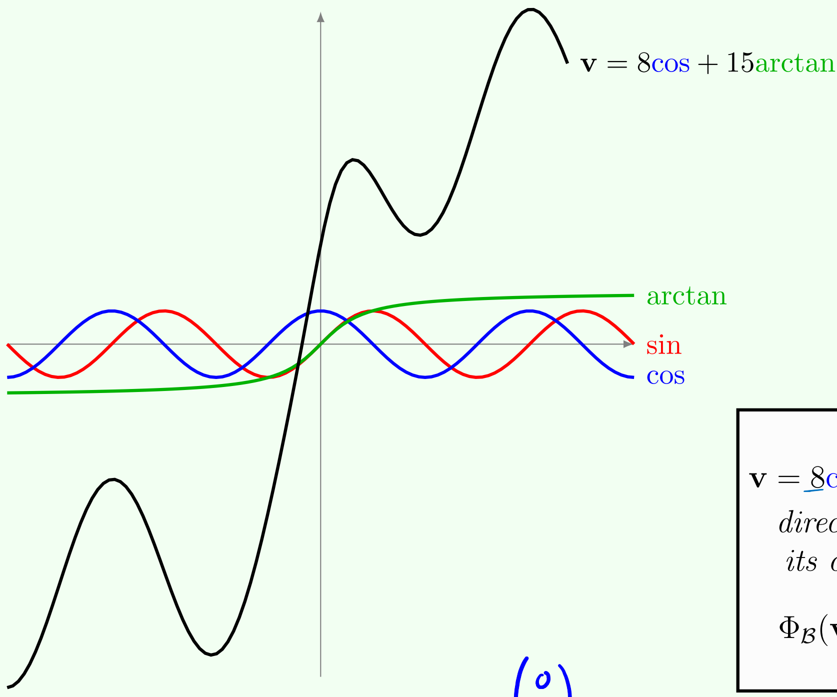
In the same manner as before, we can show that the three functions are linearly independent. Hence, they form a basis of V*:*

$$\mathcal{B} := (\sin, \cos, \arctan).$$

Now, if we look at another function $\mathbf{v} \in V$ *given by*

$$\mathbf{v}(x) = 8\cos(x) + 15\arctan(x) \quad \textit{for all} \quad x \in \mathbb{R}, \qquad \textit{hence} \quad \mathbf{v} = 8\cos + 15\arctan.$$

Then:



We find
$\mathbf{v} = 8\cos + 15\arctan$
directly by using its coordinates:

$$\Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} 0 \\ 8 \\ 15 \end{pmatrix}.$$

$$\Phi_{\mathcal{B}}(1 \cdot \cos + 3\cos) = \Phi_{\mathcal{B}}(4\cos) = 4 \cdot \Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\boxed{\Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2}$$

Rule of thumb: V is completely represented by $\mathbb{F}^n$

Each $\mathbb{F}$*-vector space* V *with* $n := \dim(V) < \infty$ *is represented by* $\mathbb{F}^n$ *if you fix a basis* $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$

For each vector $\mathbf{v} \in V$, *there is exactly one coordinate vector* $\Phi_{\mathcal{B}}(\mathbf{v}) \in \mathbb{F}^n$. *Instead of using* $\mathbf{v} \in V$, *one can also do calculations with* $\Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n$.

$\hookrightarrow \Phi_{\mathcal{B}} : V \to \mathbb{F}^n$ invertible linear map

V: for proofs

$\mathbb{F}^n$: for calculations / computer

Example 7.17. The polynomials $\mathbf{p}, \mathbf{q} \in \mathcal{P}_3(\mathbb{R})$ given by $\mathbf{p}(x) = 2x^3 - x^2 + 7$ and $\mathbf{q}(x) = x^2 + 3$ can be represented with the monomial basis $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ by the coordinate vectors:

$$\Phi_{\mathcal{B}} : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4 , \quad \Phi_{\mathcal{B}}(m_{j-1}) = e_j \quad \text{for all } j = 1, \dots, 4$$

$$\Phi_{\mathcal{B}}(p) = \begin{pmatrix} 7 \\ 0 \\ -1 \\ 2 \end{pmatrix} , \quad \Phi_{\mathcal{B}}(q) = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Phi(2q) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

Example 7.18. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ has the following coordinate vector with respect to the basis $\mathcal{B}$ from equation (7.1):

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$\Phi_{\mathcal{B}}(A) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} .$$

The matrix $3A$ has the coordinate vector $\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$.

The matrix

$$C = \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

$$U = \left\{ \begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

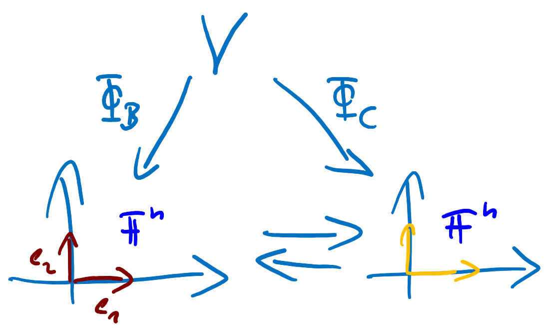
$$\Phi_{\mathcal{B}} : U \to \mathbb{R}^2$$

$$\Phi_{\mathcal{B}}(C) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\Phi_{\mathcal{B}}(4C) = 4 \cdot \Phi_{\mathcal{B}}(C) = \begin{pmatrix} 20 \\ 8 \end{pmatrix}$$

7.4.2 Change of basis

Choosing different bases?



For a given basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of the vector space V, we have the linear map

$$\boxed{\Phi_{\mathcal{B}} : V \to \mathbb{F}^n , \quad \Phi_{\mathcal{B}}(\mathbf{b}_j) = \mathbf{e}_j \quad \textit{for all } j}$$

which is also invertible. We have called it the *basis isomorphism*. For a given element $\mathbf{v} = \alpha_1\mathbf{b}_1 + \ldots + \alpha_n\mathbf{b}_n$, we can write:

$$\mathbf{v} = \alpha_1\mathbf{b}_1 + \ldots + \alpha_n\mathbf{b}_n = \alpha_1\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_1) + \ldots + \alpha_n\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_n) = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{v}))\,.$$

coordinate vector

$$\Phi_{\mathcal{B}}^{-1} : \mathbb{F}^n \to V\,, \qquad \Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j) = \mathbf{b}_j \quad \textit{for all } j$$

Example 7.19. Consider the already introduced monomial basis $\mathcal{B} = (\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) = (x \mapsto x^2, x \mapsto x, x \mapsto 1)$ of the space $\mathcal{P}_2(\mathbb{R})$ and the polynomial $\mathbf{p} \in \mathcal{P}_2(\mathbb{R})$ defined by $\mathbf{p}(x) = 4x^2 + 3x - 2$. Then:

$$\mathbf{p} = \Phi_{\mathcal{B}}^{-1}\begin{pmatrix}4\\3\\-2\end{pmatrix} = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{p})), \qquad \text{since} \qquad \mathbf{p} = 4\mathbf{m}_2 + 3\mathbf{m}_1 - 2\mathbf{m}_0.$$

Example 7.20. Let $V = \mathbb{R}^2$ and choose the basis $\mathcal{B} = (\binom{1}{1}, \binom{0}{1})$. The vector $\mathbf{x} = \binom{3}{7} \in V$ can then be represented as:

$$\mathbf{x} = \begin{pmatrix}3\\7\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{x}))$$

$\Phi_B : \mathbb{R}^2 \to \mathbb{R}^2$

$\binom{1}{1} \mapsto \binom{1}{0}$, $\Phi_B(x) = \binom{3}{4}$, $\Phi_B^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ has a corresponding matrix $A = \begin{pmatrix}1&0\\1&1\end{pmatrix}$

$\binom{0}{1} \mapsto \binom{0}{1}$

$f_A = \Phi_B^{-1}$

Now, let $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ be another basis of V.

For $v \in V$: $v = \alpha_1 b_1 + \ldots + \alpha_n b_n$

$v = \alpha_1' c_1 + \ldots + \alpha_n' \cdot c_n$

Question: Old versus new coordinates

How to switch between both coordinate vectors?

$\mathcal{B}$-coordinates $\Phi_{\mathcal{B}}(\mathbf{v}) = \begin{pmatrix}\alpha_1\\\vdots\\\alpha_n\end{pmatrix} \quad \overset{?}{\longleftrightarrow} \quad$ *$\mathcal{C}$-coordinates* $\Phi_{\mathcal{C}}(\mathbf{v}) = \begin{pmatrix}\alpha_1'\\\vdots\\\alpha_n'\end{pmatrix}$

$$\begin{tikzcd} & V & \\ \mathbb{F}^n \arrow[ur, "\Phi_{\mathcal{B}}^{-1}"] \arrow[rr, shift left, "?"] & & \mathbb{F}^n \arrow[ul, "\Phi_{\mathcal{C}}^{-1}"'] \arrow[ll, shift left, "?"] \end{tikzcd}$$

$f: \begin{pmatrix}\alpha_1\\\vdots\\\alpha_n\end{pmatrix} \longmapsto \begin{pmatrix}\alpha_1'\\\vdots\\\alpha_n'\end{pmatrix}$ change of basis without calculations inside of V.

$f : \mathbb{F}^n \to \mathbb{F}^n$ linear! First $\Phi_{\mathcal{B}}^{-1}$ then $\Phi_{\mathcal{C}}$

To get the map from "left to right" by $f : \mathbb{F}^n \to \mathbb{F}^n$, $f := \Phi_{\mathcal{C}} \circ \Phi_{\mathcal{B}}^{-1}$. More concretely for all canonical unit vectors $\mathbf{e}_j \in \mathbb{F}^n$, we get:

$$f(\mathbf{e}_j) = \Phi_{\mathcal{C}}(\Phi_{\mathcal{B}}^{-1}(\mathbf{e}_j)) = \Phi_{\mathcal{C}}(\mathbf{b}_j) \tag{7.7}$$

Since f is a linear map, we find a uniquely determined matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. This matrix is determined by equation 7.7 and given a suitable name:

Transformation matrix

$$T_{\mathcal{C}\leftarrow\mathcal{B}} := \begin{pmatrix} | & & | \\ \Phi_{\mathcal{C}}(\mathbf{b}_1) & \cdots & \Phi_{\mathcal{C}}(\mathbf{b}_n) \\ | & & | \end{pmatrix} \in \mathbb{F}^{n\times n} \tag{7.8}$$

The corresponding linear map gives us a sense of switching from basis $\mathcal{B}$ to the basis $\mathcal{C}$. Also a good mnemonic is:

$\in V$

$$\Phi_{\mathcal{C}}^{-1} T_{\mathcal{C}\leftarrow\mathcal{B}}\, \mathbf{x} = \Phi_{\mathcal{B}}^{-1}\, \mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{F}^n \tag{7.9}$$

Now, if we have a vector $\mathbf{v} \in V$ and its coordinate vector $\Phi_{\mathcal{B}}(\mathbf{v})$ and $\Phi_{\mathcal{C}}(\mathbf{v})$, respectively, then we can calculate:

$$T_{\mathcal{C}\leftarrow\mathcal{B}} \Phi_{\mathcal{B}}(\mathbf{v}) = \Phi_{\mathcal{C}}(\Phi_{\mathcal{B}}^{-1}(\Phi_{\mathcal{B}}(\mathbf{v}))) = \Phi_{\mathcal{C}}(\mathbf{v}).$$

We fix our result:

Transformation formula

$$\Phi_{\mathcal{C}}(\mathbf{v}) \;=\; T_{\mathcal{C}\leftarrow\mathcal{B}}\, \Phi_{\mathcal{B}}(\mathbf{v}). \tag{7.10}$$

$$T_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{pmatrix} | & & | \\ \Phi_{\mathcal{B}}(c_1) & \cdots & \Phi_{\mathcal{B}}(c_n) \\ | & & | \end{pmatrix}$$

$$\begin{tikzcd}
 & \mathbf{v} \in V & \\
\Phi_{\mathcal{B}}(\mathbf{v}) \in \mathbb{F}^n \arrow[ur, "\Phi_{\mathcal{B}}^{-1}"] \arrow[rr, shift left, "T_{\mathcal{C}\leftarrow\mathcal{B}}"] & & \Phi_{\mathcal{C}}(\mathbf{v}) \in \mathbb{F}^n \arrow[ul, "\Phi_{\mathcal{C}}^{-1}"'] \arrow[ll, shift left, "T_{\mathcal{B}\leftarrow\mathcal{C}}"]
\end{tikzcd}$$

$$\Phi_{\mathcal{B}}(v) = T_{\mathcal{B}\leftarrow\mathcal{C}}\, \Phi_{\mathcal{C}}(v)$$

$$T_{\mathcal{B}\leftarrow\mathcal{C}} \;=\; (T_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}. \tag{7.11}$$

You should know how to calculate inverses!

Rule of thumb: How to get the transformation matrix $T_{\mathcal{C}\leftarrow\mathcal{B}}$

The notation $T_{\mathcal{C}\leftarrow\mathcal{B}}$ means: We put the vector in $\mathcal{B}$-coordinates in (from the right) and get out the vector in $\mathcal{C}$-coordinates. To get the transformation matrix $T_{\mathcal{C}\leftarrow\mathcal{B}}$ write the basis vectors of $\mathcal{B}$ in $\mathcal{C}$-coordinates and put them as columns in a matrix.

Example 7.21. We already know the ~~poly~~mon nomial basis

$$\mathcal{B} = (\underbrace{\mathbf{m}_2}_{=:\,\mathbf{b}_1}, \underbrace{\mathbf{m}_1}_{=:\,\mathbf{b}_2}, \underbrace{\mathbf{m}_0}_{=:\,\mathbf{b}_3}) = (x \mapsto x^2, x \mapsto x, x \mapsto 1)$$

$$\begin{tikzcd} & \mathcal{P}_2(\mathbb{R}) \arrow[dl, "\Phi_{\mathcal{B}}"'] \arrow[dr, "\Phi_{\mathcal{C}}"] & \\ \mathbb{R}^3 \arrow[rr, shift left] & & \mathbb{R}^3 \arrow[ll, shift left, "T_{\mathcal{B}\leftarrow\mathcal{C}}"] \end{tikzcd}$$

in $\mathcal{P}_2(\mathbb{R})$. Now, we can easily show that

$$\mathcal{C} = (\underbrace{\mathbf{m}_2 - \tfrac{1}{2}\mathbf{m}_1}_{=:\,\mathbf{c}_1}, \underbrace{\mathbf{m}_2 + \tfrac{1}{2}\mathbf{m}_1}_{=:\,\mathbf{c}_2}, \underbrace{\mathbf{m}_0}_{=:\,\mathbf{c}_3}).$$

defines also a basis of $\mathcal{P}_2(\mathbb{R})$. Now we know how to change between these two bases. Therefore, we calculate the transformation matrices. The first thing you should note is that the basis $\mathcal{C}$ is already given in linear combinations of the basis vectors from $\mathcal{B}$. Hence we get:

$$\Phi_{\mathcal{B}}(c_1) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \quad \Phi_{\mathcal{B}}(c_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad \Phi_{\mathcal{B}}(c_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \Phi_{\mathcal{B}}(\Phi_{\mathcal{C}}^{-1}(e_1))$$

$$\Longrightarrow \quad T_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By calculating the inverse, we get the other transformation matrix $T_{\mathcal{C}\leftarrow\mathcal{B}} = (T_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$:

Calculation!

$$T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | & | \\ \Phi_{\mathcal{C}}(\mathbf{b}_1) & \Phi_{\mathcal{C}}(\mathbf{b}_2) & \Phi_{\mathcal{C}}(\mathbf{b}_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For an arbitrary polynomial $\mathbf{p}(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, we get:

$$\Phi_{\mathcal{C}}(\mathbf{p}) = T_{\mathcal{C}\leftarrow\mathcal{B}}\,\Phi_{\mathcal{B}}(\mathbf{p}) = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a/2 - b \\ a/2 + b \\ c \end{pmatrix}.$$

Hence, for our example $\mathbf{p}(x) = 4x^2 + 3x - 2$, $a = 4$, $b = 3$ and $c = -2$, we get:

$$\Phi_\mathcal{C}(\mathbf{p}) = \begin{pmatrix} 4/2 - 3 \\ 4/2 + 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}.$$

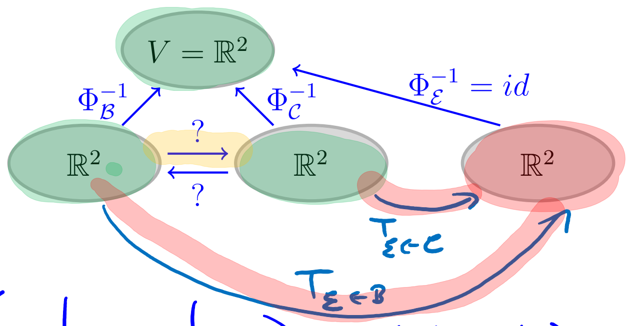
Let us check again if this was all correct:

$$(-1)\underbrace{(x^2 - \tfrac{1}{2}x)}_{\mathbf{c}_1(x)} + 5\underbrace{(x^2 + \tfrac{1}{2}x)}_{\mathbf{c}_2(x)} + (-2)\underbrace{1}_{\mathbf{c}_3(x)} = (-1+5)x^2 + (\tfrac{1}{2} + \tfrac{5}{2})x + (-2)1 = 4x^2 + 3x - 2.$$

Example 7.22. Now, we look at $\mathbb{R}^2$ with the two bases $\mathcal{B} = \left(\binom{1}{2}, \binom{3}{4}\right)$ and $\mathcal{C} = \left(\binom{1}{0}, \binom{2}{2}\right)$.

$T_{B\leftarrow C}$, $T_{C\leftarrow B}$ are not immediately given

⇝ include basis $\mathcal{E} = \left(\underbrace{\binom{1}{0}}_{=e_1}, \underbrace{\binom{0}{1}}_{=e_2}\right)$



Idea: $T_{\mathcal{E}\leftarrow B} = \begin{pmatrix} | & | \\ \Phi_\mathcal{E}(b_1) & \Phi_\mathcal{E}(b_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix}$

$$T_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad T_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}.$$

Now, for getting $T_{\mathcal{C}\leftarrow\mathcal{B}}$, we have to combine:

$$T_{\mathcal{C}\leftarrow\mathcal{B}} = T_{\mathcal{C}\leftarrow\mathcal{E}} T_{\mathcal{E}\leftarrow\mathcal{B}} = (T_{\mathcal{E}\leftarrow\mathcal{C}})^{-1} T_{\mathcal{E}\leftarrow\mathcal{B}}.$$

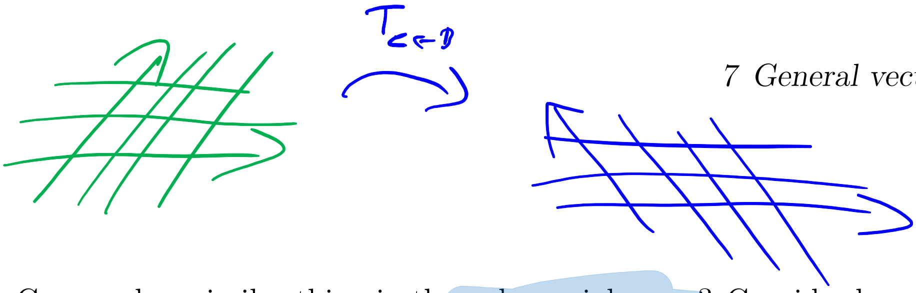
$$= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right)$$

$$(T_{\mathcal{E}\leftarrow\mathcal{C}} \,|\, T_{\mathcal{E}\leftarrow\mathcal{B}}) \rightsquigarrow (\mathbb{1} \,|\, T_{\mathcal{C}\leftarrow\mathcal{B}}), \qquad \text{so} \qquad \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Gauß-Jordan algorithm

⇝ $T_{\mathcal{C}\leftarrow B}$

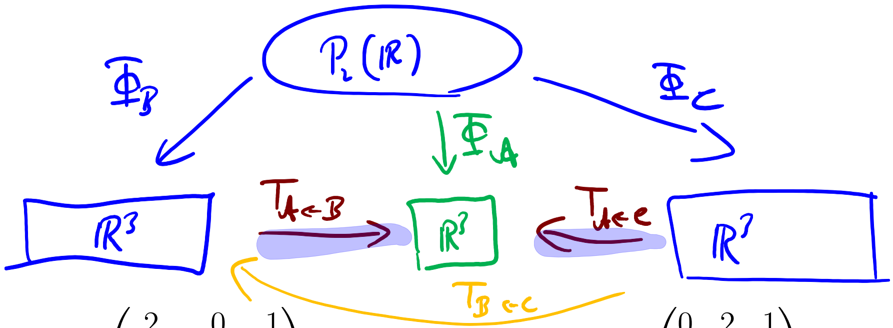


Question: Can we do a similar thing in the polynomial space? Consider bases $\mathcal{B}$ and $\mathcal{C}$ that is not the simple monomial basis:

$$\mathcal{B} = (\underbrace{2\mathbf{m}_2 - 1\mathbf{m}_1}_{=:\mathbf{b}_1},\ \underbrace{-8\mathbf{m}_1 - 2\mathbf{m}_0}_{=:\mathbf{b}_2},\ \underbrace{1\mathbf{m}_2 + 4\mathbf{m}_1 + 1\mathbf{m}_0}_{=:\mathbf{b}_3})$$

$$\text{and} \quad \mathcal{C} = (\underbrace{1\mathbf{m}_1 + 1\mathbf{m}_0}_{=:\mathbf{c}_1},\ \underbrace{2\mathbf{m}_2 + 2\mathbf{m}_1}_{=:\mathbf{c}_2},\ \underbrace{1\mathbf{m}_2 + 1\mathbf{m}_0}_{=:\mathbf{c}_3}).$$

Answer: Yes, we can do the same by adding the the monomial basis (or a other well-known basis) in the middle. We call the monomial basis by $\mathcal{A}$, which means $\mathcal{A} = (\mathbf{m}_2,\ \mathbf{m}_1,\ \mathbf{m}_0)$. Then $T_{\mathcal{A}\leftarrow\mathcal{B}}$ and $T_{\mathcal{A}\leftarrow\mathcal{C}}$ are immediately given:



$$T_{\mathcal{A}\leftarrow\mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -8 & 4 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathcal{A}\leftarrow\mathcal{C}} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and then we get $T_{\mathcal{B}\leftarrow\mathcal{C}}$:

$T_{\mathcal{B}\leftarrow\mathcal{C}}$ by using an additional "nice" basis $\mathcal{A}$

$$T_{\mathcal{A}\leftarrow\mathcal{C}} = T_{\mathcal{A}\leftarrow\mathcal{B}}\, T_{\mathcal{B}\leftarrow\mathcal{C}}$$

$$\Phi_{\mathcal{A}}(\mathbf{x}) \xleftarrow{T_{\mathcal{A}\leftarrow\mathcal{B}}} \Phi_{\mathcal{B}}(\mathbf{x}) \xleftarrow{T_{\mathcal{B}\leftarrow\mathcal{C}}} \Phi_{\mathcal{C}}(\mathbf{x})$$

and hence $$T_{\mathcal{B}\leftarrow\mathcal{C}} = (T_{\mathcal{A}\leftarrow\mathcal{B}})^{-1}\, T_{\mathcal{A}\leftarrow\mathcal{C}}.$$

Since we again have to find an inverse of a matrix, we can use the Gauß-Jordan algorithm again:

$$(T_{\mathcal{A}\leftarrow\mathcal{B}} \mid T_{\mathcal{A}\leftarrow\mathcal{C}}) \rightsquigarrow (\mathbb{1} \mid T_{\mathcal{B}\leftarrow\mathcal{C}}). \tag{7.12}$$

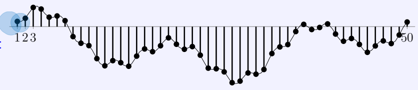
For our example, this gives us:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 2 & 1 \\ -1 & -8 & 4 & 1 & 2 & 0 \\ 0 & -2 & 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \boxed{\begin{array}{ccc} 3 & -2 & 4 \\ -7/2 & 3 & -4 \\ -6 & 6 & -7 \end{array}} \\ 0 & 0 & 1 & \end{array}\right).$$

The boxed matrix is indeed $T_{\mathcal{B}\leftarrow\mathcal{C}}$.

Change of basis for audio: WAV vs. MP3

Assume you have an audio signal $\mathbf{f}$ *given at finite time steps* $t = 1, 2, ..., 50$ *(e.g. milliseconds). Hence, you have some measure values* $f_1, f_2, \ldots, f_{50} \in \mathbb{R}$.

$\mathbf{f} =$ 

$\mathbf{b}_1$: 

$\mathbf{b}_2$: 

$\vdots$

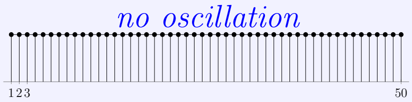
$\mathbf{b}_{50}$: 

At this point you know that the audio signal $\mathbf{f}$ *is a vector in a 50-dimensional space, which can be represented with respect to the canonical basis* $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_{50})$ *of* $\mathbb{R}^{50}$*. The coordinates of* $\mathbf{f}$ *are exactly the values* $f_1, f_2, \ldots, f_{50}$*. For describing tones (so oscillations) this basis is not optimal!*

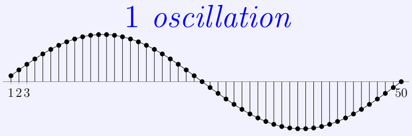
We want to change to a basis $\mathcal{C}$ *of* $\mathbb{R}^{50}$*, which is better fitting for tones.*

Sine waves | ***Cosine waves***

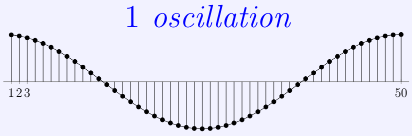
no oscillation

$\mathbf{c}_{25}$: 

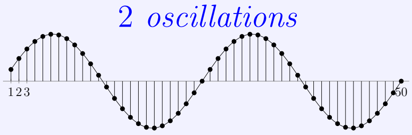
1 oscillation

$\mathbf{c}_1$: 

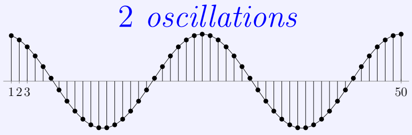
1 oscillation

$\mathbf{c}_{26}$: 

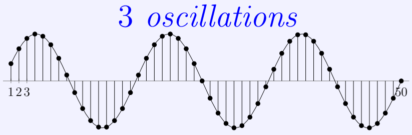
2 oscillations

$\mathbf{c}_2$: 

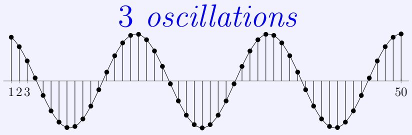
2 oscillations

$\mathbf{c}_{27}$: 

3 oscillations

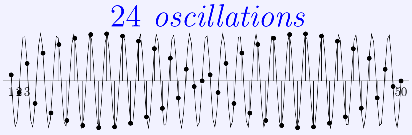
$\mathbf{c}_3$: 

3 oscillations

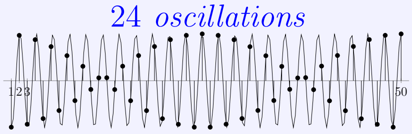
$\mathbf{c}_{28}$: 

$\vdots$ $\vdots$

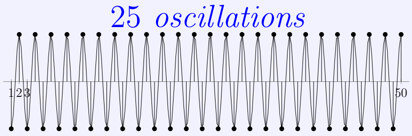
24 oscillations

$\mathbf{c}_{24}$: 

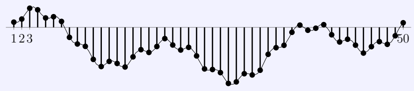
24 oscillations

$\mathbf{c}_{49}$: 

25 oscillations

$\mathbf{c}_{50}$: 

One can show: $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{50})$ *is also linearly independent and hence a basis of* $\mathbb{R}^{50}$*.*

$\mathbf{f} =$ 

The signal $\mathbf{f}$ *from above has the following form:*

$$\mathbf{f} = \mathbf{c}_3 - \mathbf{c}_{25} + 3\mathbf{c}_{26} + \tfrac{1}{4}\mathbf{c}_{40}.$$

We reckon that most signals $\mathbf{f}$ *are a superposition of some "basic tones"* $\mathbf{c}_i$*.*

Compression: One stores only the coordinates in $\Phi_{\mathcal{C}}(\mathbf{f})$*. One can also focus on the (for humans) important frequencies and ignore the higher and lower ones (e.g. MP3 file format). All this saves storage space instead of storing the coordinates* $\Phi_{\mathcal{B}}(\mathbf{f}) = (f_1, \dots, f_{50})^T$ *(e.g. WAV file format). Similar ideas exist for two-dimensional signals like pictures:* $\Rightarrow$ *BMP vs. JPG.*

Information: The change of basis from $\mathcal{B}$ *to* $\mathcal{C}$ *is important for a lot of applications and known as the Fourier transform. We will consider it in more detail in the analysis lecture.*

7.5 General vector space with inner product and norms

Recall that in the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, besides the algebraic structure given by

vector addition $+$ and the scalar multiplication $\cdot$,

we also defined a geometric structure by choosing

an inner product $\langle \cdot, \cdot \rangle$ and also a norm $\|\cdot\|$

for measuring angles and lengths.

Attention! Convention for $\mathbb{F}=\mathbb{R}$ and $\mathbb{F}=\mathbb{C}$

Since we handle the cases $\mathbb{F} = \mathbb{R}$ *and* $\mathbb{F} = \mathbb{C}$ *simultaneously, we also use the notion of the complex conjugation in the real case. Hence, for* $\alpha \in \mathbb{F}$ *we write:*

$$\overline{\alpha} := \begin{cases} \alpha & \text{if } \mathbb{F} = \mathbb{R}, \\ \overline{\alpha} & \text{if } \mathbb{F} = \mathbb{C} \quad \text{(complex conjugate number).} \end{cases}$$

Analogously, for a matrix $A \in \mathbb{F}^{m \times n}$ *with* $m, n \in \mathbb{N}$*:*

$$A^* := \begin{cases} A^T & \text{if } \mathbb{F} = \mathbb{R} \quad \text{(transpose)}, \\ A^* & \text{if } \mathbb{F} = \mathbb{C} \quad \text{(adjoint)}. \end{cases}$$