7

General vector spaces

7.1 Vector space in its full glory

$\mathbb{R}^n$ (addition, scaling), $\mathbb{R}^{m\times n}$, $\mathbb{C}^n$, $\mathbb{C}^{m\times n}$

$\mathbb{R}, \mathbb{C} \rightsquigarrow$ field (calculation for numbers), $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

Definition 7.1. Real or complex vector spaces

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. A nonempty set V together with two operations,

- *a vector addition $+ : V \times V \to V$,*
- *and a scalar multiplication $\cdot : \mathbb{F} \times V \to V$,*

where the rules below are satisfied, is called an $\mathbb{F}$-vector space. The elements of V are called vectors, and the elements $\mathbb{F}$ are called scalars. The two operations have to satisfy the following rules:

(1) $\forall \mathbf{v}, \mathbf{w} \in V: \quad \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ *(+ is commutative)*

(2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V: \quad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *(+ is associative)*

(3) *There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ we have $\mathbf{v} + \mathbf{o} = \mathbf{v}$.*

(4) *For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.*

(5) *For the number $1 \in \mathbb{F}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.*

(6) $\forall \alpha, \beta \in \mathbb{F} \quad \forall \mathbf{v} \in V: \quad \lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v}$ *($\cdot$ is associative)*

(7) $\forall \lambda \in \mathbb{F} \quad \forall \mathbf{v}, \mathbf{w} \in V: \quad \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w})$ *(distributive $\cdot +$)*

(8) $\forall \lambda, \mu \in \mathbb{F} \quad \forall \mathbf{v} \in V: \quad (\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v})$ *(distributive $+ \cdot$)*

$\hookrightarrow$ abstract vector space (visualization)

Example 7.2. $\mathbb{R}^n$ and $\mathbb{C}^n$. At this point, we are very familiar with the space $\mathbb{F}^n$, where the vectors have n components consisting of numbers from $\mathbb{F}$ and the addition and scalar multiplication is done componentwise:

$$\lambda \in \mathbb{F},\ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

Now, this is now just a special case of an $\mathbb{F}$-vector space. ✓

$\mathbb{F}^n$ vectors $\rightsquigarrow$ abstract vectors

Example 7.3. Matrices. The set of matrices $V := \mathbb{F}^{m \times n}$ together with the matrix addition and scalar multiplication

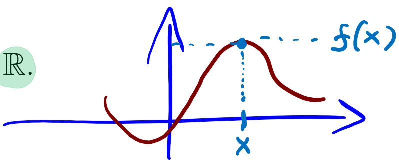
$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}}_{B} := \underbrace{\begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}}_{A+B}.$$

$$\lambda \cdot \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} := \underbrace{\begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{pmatrix}}_{\lambda \cdot A}.$$

defines also an $\mathbb{F}$-vector space.

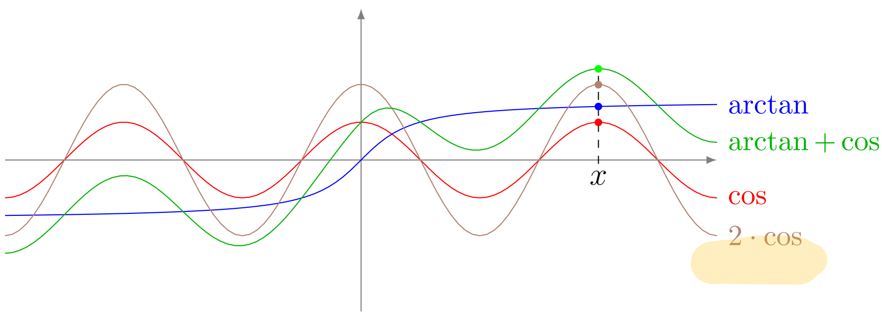
matrices are abstract vectors

Example 7.4. Functions. Let $\mathcal{F}(\mathbb{R})$ be the set of functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$. For all $\alpha \in \mathbb{R}$ and $\mathbf{f}, \mathbf{g} \in \mathcal{F}(\mathbb{R})$ define $\alpha \cdot \mathbf{f}$ and $\mathbf{f} + \mathbf{g}$ by



$$\begin{aligned} (\alpha \cdot \mathbf{f})(x) &:= \alpha \cdot \mathbf{f}(x) \\ (\mathbf{f} + \mathbf{g})(x) &:= \mathbf{f}(x) + \mathbf{g}(x) \end{aligned}$$

for all $x \in \mathbb{R}$.



This is a natural definition for the α-multiple of a function and the sum of two functions.

$\alpha f : \mathbb{R} \to \mathbb{R}$ (same domain and codomain)

$f + g : \mathbb{R} \to \mathbb{R}$ (" " " ")

Hence, $\mathcal{F}(\mathbb{R})$ with $+$ and $\cdot$ is an ~~$\mathbb{F}$~~ $\mathbb{R}$-vector space.

Check rules (1)-(8)

$\mathbb{R}^n$: $o = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Question for you: What is zero vector?

$o : \mathbb{R} \to \mathbb{R}$

$o(x) = 0$

zero vector

zero in $\mathbb{R}$

Lemma 7.5. o=0 · f and −f=(−1) · f

Let V be an $\mathbb{F}$-vector space with the operations $+$ and $\cdot$. For all $\mathbf{f} \in V$, we have

$$\mathbf{o} = 0 \cdot \mathbf{f} \quad \text{and} \quad -\mathbf{f} = (-1) \cdot \mathbf{f}.$$

Proof: $0 \cdot f = (0+0) \cdot f \overset{(8)}{=} 0 \cdot f + 0 \cdot f$

$\overset{(4)}{\Rightarrow} \quad 0 \cdot f + (-0 \cdot f) = 0 \cdot f + \big(\underbrace{0 \cdot f + (-0 \cdot f)}_{=o}\big) \overset{(2),(4)}{=} 0 \cdot f + o$

$\quad /\!/(4) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \backslash\backslash(3)$

$\quad o \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \cdot f$ ✓

$o = 0 \cdot f = (1 + (-1)) \cdot f \overset{(8)}{=} 1 \cdot f + (-1) \cdot f \overset{(5)}{=} f + (-1) \cdot f$

$\overset{(4)}{\Rightarrow} \quad o + (-f) = f + (-1) f + (-f) \overset{(1),(2)}{=} \underbrace{f + (-f)}_{=o \ (4)} + (-1) f \overset{(3)}{=} (-1) \cdot f$

$\quad /\!/(3)$

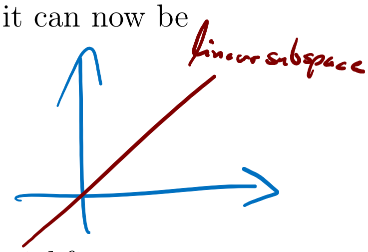
$\quad -f$

7.2 Linear subspaces

We already defined the term *linear subspace* in $\mathbb{R}^n$ and $\mathbb{C}^n$ and, of course, it can now be generalised for the general vector spaces.

Let us look at a special subset from Example 7.4:

Vector space inside another one



Example 7.6. Polynomial functions. Let $\mathcal{P}(\mathbb{R})$ denote the set of polynomial functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$. We know that $\mathcal{P}(\mathbb{R})$ is a nonempty subset of $\mathcal{F}(\mathbb{R})$ (set of all functions $\mathbb{R} \to \mathbb{R}$) from Example 7.4. The addition $+$ and scalar multiplication $\cdot$ are just inherited from $\mathcal{F}(\mathbb{R})$.

$f(x) = 2 \cdot x^2 + 1x$ a polynomial

$f(x) = \cos(x)$ not a polynomial

Is $\mathcal{P}(\mathbb{R})$ also a vector space?

Need addition: $+ : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$

Need scalar mult. $\cdot : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$

In other words: Is $+, \cdot$ well-defined on $\mathcal{P}(\mathbb{R})$?

To show: $f, g \in \mathcal{P}(\mathbb{R}),\ \alpha \in \mathbb{R} \Rightarrow f+g \in \mathcal{P}(\mathbb{R}),\ \alpha \cdot f \in \mathcal{P}(\mathbb{R})$ $(*)$

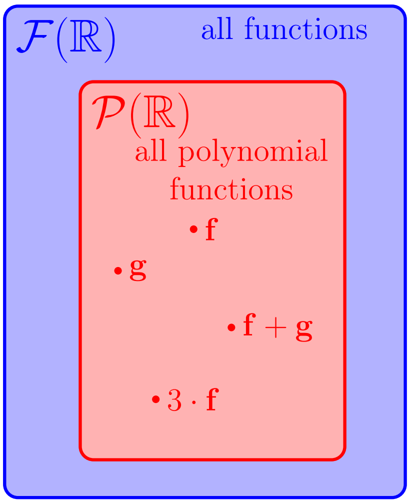
(closed under $+$ and $\cdot$)

Are (1) – (8) satisfied?

Now checking (1)–(8) is very fast because:

(1) $\mathbf{f}+\mathbf{g}=\mathbf{g}+\mathbf{f}$,
(2) $\mathbf{f}+(\mathbf{g}+\mathbf{h})=(\mathbf{f}+\mathbf{g})+\mathbf{h}$,
(5) $1\cdot\mathbf{f}=\mathbf{f}$,
(6) $\alpha\cdot(\beta\cdot\mathbf{f}) = (\alpha\beta)\cdot\mathbf{f}$,
(7) $\alpha\cdot(\mathbf{f}+\mathbf{g}) = (\alpha\cdot\mathbf{f})+(\alpha\cdot\mathbf{g})$,
(8) $(\alpha+\beta)\cdot\mathbf{f} = (\alpha\cdot\mathbf{f})+(\beta\cdot\mathbf{f})$

hold for all $f \in \mathcal{F}(\mathbb{R})$. Therefore for all $f \in \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$



$\mathcal{P}(\mathbb{R})$ inherits the rules from $\mathcal{F}(\mathbb{R})$.

(3) $0 \in \mathcal{P}(\mathbb{R})$
(4) $f \in \mathcal{P}(\mathbb{R}) \Rightarrow -f \in \mathcal{P}(\mathbb{R})$

follow from $(*)$ $\Rightarrow$ $\mathcal{P}(\mathbb{R})$ vector space

U is again a vector space

Proposition & Definition 7.7. Linear subspace

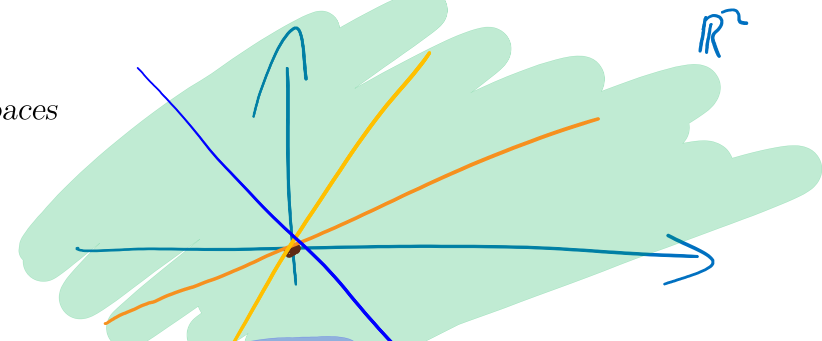
Let V be an $\mathbb{F}$-vector space and let U be a non-empty subset of V, which is closed under vector addition and scalar multiplication of V, which means

(a) for all $\mathbf{u},\mathbf{v} \in U$, we have $\mathbf{u}+\mathbf{v} \in U$ and

(b) for all $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$, we have $\alpha\cdot\mathbf{u} \in U$.

Then U is also an $\mathbb{F}$-vector space. In this case, U is called a linear subspace of V or in short a subspace of V

trivial subspaces: $U = \{0\}$, $U = V$



Example 7.8. Quadratic polynomials. Let $\mathcal{P}_2(\mathbb{R})$ be the set of all polynomials with degree ≤ 2, which means

$$\text{all functions} \quad \mathbf{p} : \mathbb{R} \to \mathbb{R}, \quad \mathbf{p}(x) = a_2x^2 + a_1x + a_0 \quad \text{with} \quad a_2, a_1, a_0 \in \mathbb{R}.$$

Is $\mathcal{P}_2(\mathbb{R})$ with the vector addition $+$ and $\cdot$ from $\mathcal{F}(\mathbb{R})$ a vector space?

Obviously, $\mathcal{P}_2(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R}) \neq \emptyset$. Using Proposition 7.7 we only have to check that $\mathcal{P}_2(\mathbb{R})$ is closed under $+$ and $\cdot$, which means that we have to check (a) and (b):

Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then, there are $a_2, a_1, a_0, b_2, b_1, b_0 \in \mathbb{R}$ such that

$$\mathbf{p}(x) = a_2x^2 + a_1x + a_0 \quad \text{and} \quad \mathbf{q}(x) = b_2x^2 + b_1x + b_0.$$

Hence:

$$\begin{aligned}
(\mathbf{p}+\mathbf{q})(x) &= \mathbf{p}(x) + \mathbf{q}(x) &&= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\
& &&= (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0), \quad \in \mathcal{P}_2(\mathbb{R}) \\
(\alpha \cdot \mathbf{p})(x) &= \alpha \cdot \mathbf{p}(x) &&= \alpha \cdot (a_2x^2 + a_1x + a_0) \\
& &&= (\alpha a_2)x^2 + (\alpha a_1)x + (\alpha a_0)
\end{aligned}$$

We conclude that $\mathbf{p}+\mathbf{q} \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \cdot \mathbf{p} \in \mathcal{P}_2(\mathbb{R})$. The set $\mathcal{P}_2(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$ and a vector space for its own.

Analogously, for $n \in \mathbb{N}_0$, we define $\mathcal{P}_n(\mathbb{R})$ as the set of all polynomials with degree $\leq n$. It forms again a vector space with the operations $+$ and $\cdot$ from $\mathcal{F}$.

↳ Same proof!

Example 7.9. Upper triangular matrices Let $n \in \mathbb{N}$ and ◥$\mathbb{R}^{n\times n}$ the set of all upper triangular matrices $A \in \mathbb{R}^{n\times n}$. The operations $+$ and $\cdot$ are the same as before for all matrices.

Check: $\nabla\mathbb{R}^{n\times n}$ non-empty and (a), (b). ✓

↳ $0 \in \nabla\mathbb{R}^{n\times n}$

Example 7.10. The set of all matrices U in the following form:

$$\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & a \end{pmatrix} \quad \text{with} \quad a, b \in \mathbb{C}$$

is closed under matrix addition and the multiplication with scalars $\alpha \in \mathbb{C}$. Therefore, U is a subspace of $\mathbb{C}^{3\times 3}$ and a $\mathbb{C}$-vector space for itself.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in U \checkmark$$

$$\underbrace{\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & a \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} a' & 0 & a' \\ 0 & b' & 0 \\ a' & 0 & a' \end{pmatrix}}_{\in U} = \begin{pmatrix} a+a' & 0 & a+a' \\ 0 & b+b' & 0 \\ a+a' & 0 & a+a' \end{pmatrix} \in U \checkmark$$

If we look back at the polynomial spaces, we notice that we have the following inclusions:

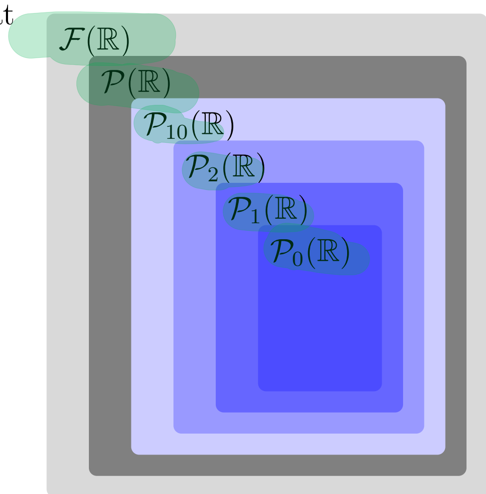
$$\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R}) \subset \mathcal{P}_2(\mathbb{R}) \subset \cdots \subset \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$$

Constant functions

sizes?

dimension

Foresight: $\dim(\mathcal{P}_n) = n+1$



7.3 Recollection: basis, dimension and other stuff

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and V be an $\mathbb{F}$-vector space with vector addition $+$ and scalar multiplication $\cdot$.

As we did for $\mathbb{R}^n$ and later for $\mathbb{C}^n$, we introduce notions like *linear independence*, *basis*, *dimension* and related definitions. In spite of considering abstract vector spaces, the notions still work exactly the same.

Definition 7.11. Same as before: Basis, dimension, and so on

Let V be an $\mathbb{F}$-vector space with operations $+$ and $\cdot$.

- *For $k \in \mathbb{N}$, vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ and scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ the vector*

$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_k \mathbf{v}_k = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i \in V$$

  is called a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

- *The set of all possible linear combinations for the vectors of a subset $M \subset V$ is called the linear hull or span of M:*

$$\mathrm{Span}\,(M) := \{\lambda_1 \mathbf{u}_1 + \cdots + \lambda_k \mathbf{u}_k \ : \ \mathbf{u}_1, \ldots, \mathbf{u}_k \in M, \ \lambda_1, \ldots, \lambda_k \in \mathbb{F}, \ k \in \mathbb{N}\} \ .$$

- *A family $\mathcal{E} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ consisting of k vectors from V is called a generating system for the subspace $U \subset V$, if $U = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.*

- *A family $\mathcal{E} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ consisting of k vectors from V is called linearly dependent if $\mathbf{o}$ can be represented by a non-trivial linear combination of vectors from $\mathcal{E}$. If there is no such non-trivial linear combination, the family is called linearly independent.*

- *A family $\mathcal{E}$ that is a generating system for $U \subset V$ and linearly independent is called a basis of U.*

- *The number of elements for a basis of U is called the dimension of U. We just write $\dim(U)$.*

Rule of thumb: Basis, dimension and similar things

- *A generating family $\mathcal{E} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of U is called this way because we can reach each point in U with linear combinations of vector from $\mathcal{E}$ and no other points.*

- *A family $\mathcal{E}$ is linear independent if we need all "family members" to span (or generate) the subspace $\mathrm{Span}(\mathcal{E})$.*

- *A basis $\mathcal{B}$ of U is a smallest generating set U. (We cannot omit a vector from B.)*

- *The dimension of a subspace U*
  - *= the number of elements of a basis of U. (All bases have the same number of elements, just redo the proof in Proposition 3.25.)*
  - *= the smallest possible size for a generating system of U. (With less vectors it is not possible to span the whole space U.)*
  - *= the maximal number of vectors from U that form a linearly independent family. (If you choose more vectors, there are always linearly dependent.)*

Example 7.12. – Matrix vector spaces

(a) The vector space $\mathbb{C}^{2\times 3}$ of all complex 2×3-matrices can be written in the following way:

$$\begin{aligned}
\mathbb{C}^{2\times 3} &= \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \varphi \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{C} \right\} \\
&= \left\{ \alpha \left(\begin{smallmatrix} 1&0&0\\0&0&0 \end{smallmatrix}\right) + \beta \left(\begin{smallmatrix} 0&1&0\\0&0&0 \end{smallmatrix}\right) + \gamma \left(\begin{smallmatrix} 0&0&1\\0&0&0 \end{smallmatrix}\right) + \delta \left(\begin{smallmatrix} 0&0&0\\1&0&0 \end{smallmatrix}\right) + \varepsilon \left(\begin{smallmatrix} 0&0&0\\0&1&0 \end{smallmatrix}\right) + \varphi \left(\begin{smallmatrix} 0&0&0\\0&0&1 \end{smallmatrix}\right) : \alpha, \beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{C} \right\} \\
&= \operatorname{Span}\left(\begin{pmatrix} 1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1 \end{pmatrix} \right)
\end{aligned}$$

Hence

$$\mathcal{B} = \left(\begin{pmatrix} 1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1 \end{pmatrix} \right)$$

is a generating system for $\mathbb{C}^{2\times 3}$. $\mathcal{B}$ is also linearly independent: From

$$\underbrace{\alpha \begin{pmatrix} 1&0&0\\0&0&0 \end{pmatrix} + \beta \begin{pmatrix} 0&1&0\\0&0&0 \end{pmatrix} + \gamma \begin{pmatrix} 0&0&1\\0&0&0 \end{pmatrix} + \delta \begin{pmatrix} 0&0&0\\1&0&0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0&0&0\\0&1&0 \end{pmatrix} + \varphi \begin{pmatrix} 0&0&0\\0&0&1 \end{pmatrix}}_{\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \varphi \end{smallmatrix}\right)} = \mathbf{o}, = \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix}$$

we conclude $\alpha = \beta = \gamma = \delta = \varepsilon = \varphi = 0$. Hence, $\mathcal{B}$ is a basis of $\mathbb{C}^{2\times 3}$ and the dimension of $\mathbb{C}^{2\times 3}$ is $|\mathcal{B}| = 6$. Analogously, one can prove: $\dim(\mathbb{F}^{m\times n}) = m \cdot n$.

(b) In a similar way, we can prove that

$$\mathcal{B} = \left(\begin{pmatrix} 1&0\\0&0 \end{pmatrix}, \begin{pmatrix} 0&1\\0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&1 \end{pmatrix} \right) \tag{7.1}$$

forms a basis of $\text{◥}\mathbb{R}^{2\times 2}$. Hence:

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} 1&0\\0&0 \end{pmatrix} + \beta \begin{pmatrix} 0&1\\0&0 \end{pmatrix} + \gamma \begin{pmatrix} 0&0\\0&1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

We conclude: $\dim(\text{◥}\mathbb{R}^{2\times 2}) = 2 + 1 = 3$.
Analogously for given $n \in \mathbb{N}$, one can prove $\dim(\text{◥}\mathbb{R}^{n\times n}) = n + (n-1) + \ldots + 1 = \frac{n(n+1)}{2}$.

(c) As a special vector space, we look at:

$$U = \left\{ \begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} = \left\{ \alpha \begin{pmatrix} 1&0&1\\0&0&0\\1&0&1 \end{pmatrix} + \beta \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

$$= \operatorname{Span}\Bigg(\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{=:A}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:B}\Bigg). \tag{7.2}$$

Hence, $\mathcal{B} := (A, B)$ is a generating system for U. Again, we show that $\mathcal{B}$ is also linearly independent. From $\alpha A + \beta B = \mathbf{o}$, one gets

$$\begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \alpha \end{pmatrix} = \alpha A + \beta B = \mathbf{o} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and concludes $\alpha = \beta = 0$. Therefore, $\mathcal{B}$ is a basis of U and $\dim(U) = 2$.

Example 7.13. – **Polynomial space $\mathcal{P}_2(\mathbb{R})$.** We define the special polynomials $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{P}_2$ by

$$\mathbf{m}_0(x) := 1, \quad \mathbf{m}_1(x) := x, \quad \text{and} \quad \mathbf{m}_2(x) := x^2 \quad \text{for all} \quad x \in \mathbb{R}$$

and see:

$$\mathcal{P}_2(\mathbb{R}) = \{x \mapsto a_2x^2 + a_1x + a_0 : a_2, a_1, a_0 \in \mathbb{R}\} = \{a_2\mathbf{m}_2 + a_1\mathbf{m}_1 + a_0\mathbf{m}_0 : a_2, a_1, a_0 \in \mathbb{R}\}$$
$$= \operatorname{Span}(\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$$

Hence, $\mathcal{B} := (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ is a generating system for $\mathcal{P}_2(\mathbb{R})$.

Remains to show: (m_0, m_1, m_2) are lin. indep.

$\alpha m_0 + \beta m_1 + \gamma m_2 = 0$ (zero vector $0: \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$)

$\Longleftrightarrow \alpha \cdot 1 + \beta x + \gamma x^2 = 0$ for all $x \in \mathbb{R}$.

$$\Rightarrow \begin{cases} x = 0: & \alpha + 0 + 0 = 0 \\ x = 1: & \alpha + \beta + \gamma = 0 \\ x = -1: & \alpha - \beta + \gamma = 0 \end{cases}$$

$\Rightarrow \alpha, \beta, \gamma$ are given by the solution set of $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right)$

$\rightarrow \alpha = \beta = \gamma = 0$ $\Rightarrow \mathcal{B}$ basis of $\mathcal{P}_2(\mathbb{R})$.

$\Rightarrow \dim(\mathcal{P}_2(\mathbb{R})) = 3$

Proposition & Definition 7.14. Monomial basis of $\mathcal{P}_n(\mathbb{R})$

Let $n \in \mathbb{N}_0$. The particular polynomials $\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n \in \mathcal{P}_n(\mathbb{R})$ defined by

$$\mathbf{m}_0(x) = 1, \quad \mathbf{m}_1(x) = x, \quad \ldots, \quad \mathbf{m}_{n-1}(x) = x^{n-1}, \quad \mathbf{m}_n(x) = x^n \quad \text{for all } x \in \mathbb{R}$$

are called monomials. The family $\mathcal{B} = (\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_n)$ forms a basis of $\mathcal{P}_n(\mathbb{R})$ and is called the monomial basis. Hence $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$.

$\rightsquigarrow$ Try a proof!

Corollary 7.15. The method of equating the coefficients

Let $\mathbf{p}$ and $\mathbf{q}$ be two real polynomials with degree $n \in \mathbb{N}$, which means

$$\mathbf{p}(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad \text{and} \qquad \mathbf{q}(x) = b_n x^n + \ldots + b_1 x + b_0$$

for some coefficients $a_n, \ldots, a_1, a_0, b_n, \ldots, b_1, b_0 \in \mathbb{R}$.
If we have the equality $\mathbf{p} = \mathbf{q}$, which means

$$a_n x^n + \ldots + a_1 x + a_0 \;=\; b_n x^n + \ldots + b_1 x + b_0, \tag{7.3}$$

for all $x \in \mathbb{R}$, then we can conclude $a_n = b_n$, $\ldots$, $a_1 = b_1$ and $a_0 = b_0$.

Remark:

Since $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$ and we have the inclusions

$$\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R}) \subset \mathcal{P}_2(\mathbb{R}) \subset \cdots \subset \mathcal{P}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}),$$

we conclude that $\dim(\mathcal{P}(\mathbb{R}))$ and $\dim(\mathcal{F}(\mathbb{R}))$ cannot be finite natural numbers. Symbolically, we write $\dim(\mathcal{P}(\mathbb{R})) = \infty$ in such a case.