Definition 6.8.

If the same eigenvalue λ appears $\alpha(\lambda)$ times in this factorisation, we say:

$$\lambda \text{ has } \underline{\text{algebraic multiplicity}}\ \alpha(\lambda)\ .$$

- If we have k *different* eigenvalues $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, then $\alpha(\lambda_1) + \cdots + \alpha(\lambda_k) = n$, because polynomials of degree n can be factorised into n linear factors.
- If λ is an eigenvalue, then $A - \lambda\mathbb{1}$ is singular, so $\gamma(\lambda) := \dim(\mathrm{Ker}(A - \lambda\mathbb{1})) \geq 1$.

eigenvalue

$$x \neq 0 \quad \Rightarrow \quad \underline{Ax = \lambda x}$$

eigenvector

Proposition 6.9. Spectrum for triangular matrices

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix.

(a) For a matrix in triangular form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix},$$

we get $\mathrm{spec}(A) = \{a_{11}, a_{22}, \dots, a_{nn}\}$*.*

(b) For a square block matrix in triangular form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

with square matrices B and D, we get $\mathrm{spec}(A) = \mathrm{spec}(B) \cup \mathrm{spec}(D)$*.*

(c) Also $\mathrm{spec}(A) = \mathrm{spec}(A^T)$*. Hence (a) and (b) also hold for lower triangular matrices.*

Proof.

$\square$

VL15 ↓

Example 6.10. We give some examples for Proposition 6.9.

(a) $\operatorname{spec}\begin{pmatrix} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & 9 \\ & & & 10 \end{pmatrix} = \{1, 5, 8, 10\}$

(b) $\operatorname{spec}\left(\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & 9 \\ \hline & & 10 & & \\ & & 11 & 12 & \\ & & 13 & 14 & 15 \end{array}\right) = \operatorname{spec}\begin{pmatrix} 1 & 2 \\ & 6 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 10 & & \\ 11 & 12 & \\ 13 & 14 & 15 \end{pmatrix} = \{1, 6, 10, 12, 15\}$

(c) $\operatorname{spec}\left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 \\ & 7 & 8 & 9 & 10 & 11 \\ \hline & & 12 & & & \\ & & 13 & 14 & & \\ \cline{3-6} & & 15 & 16 & 17 & 18 \\ & & 19 & 20 & & 21 \end{array}\right) = \operatorname{spec}\begin{pmatrix} 1 & 2 \\ & 7 \end{pmatrix} \cup \operatorname{spec}\left(\begin{array}{cc|cc} 12 & & & \\ 13 & 14 & & \\ \hline 15 & 16 & 17 & 18 \\ 19 & 20 & & 21 \end{array}\right)$

$= \operatorname{spec}\begin{pmatrix} 1 & 2 \\ & 7 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 12 & \\ 13 & 14 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 17 & 18 \\ & 21 \end{pmatrix} = \{1, 7, 12, 14, 17, 21\}$

$\rightsquigarrow$ Always try to use block decompositions

Recall: λ ev of A $\Leftrightarrow$ $\det(A - \lambda \mathbb{1}) = 0$ (zeros of the char. polynomial)

$\hookrightarrow$ Leibniz formula: $\det(A - \lambda\mathbb{1}) = \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (A - \lambda\mathbb{1})_{\sigma(i),i} = \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) \prod (a_{\sigma(i),i} - \lambda \delta_{\sigma(i),i})$

$\rightsquigarrow$ For $\lambda^n \Leftrightarrow \sigma = id$

$p_A(0) = \det(A)$

For $\lambda^{n-1} \Leftrightarrow \sigma = id$

Remark:

The characteristic polynomial for $A \in \mathbb{R}^{n\times n}$ is of the following form

$$p_A(\lambda) = (-1)^n \lambda^n + \operatorname{tr}(A)(-1)^{n-1}\lambda^{n-1} + \cdots + \det(A), \tag{6.1}$$

where $\operatorname{tr}(A) := \sum_{j=1}^{n} a_{jj}$ *is the sum of the diagonal, the so-called trace of A.*

6.3 Complex matrices and vectors

$\rightsquigarrow$ $\operatorname{spec}(A) \subset \mathbb{C}$

$\rightsquigarrow$ Consider $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$

Definition 6.11. Complex matrices

For $m, n \in \mathbb{N}$, the set of all $m \times n$ matrices with entries in $\mathbb{C}$ is denoted by $\mathbb{C}^{m\times n}$. Analogously, $\mathbb{C}^n$ denotes the set of all (column-)vectors with n entries in $\mathbb{C}$.

Addition: $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$, $\lambda \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda \cdot v_1 \\ \vdots \\ \lambda \cdot v_n \end{pmatrix}$, $v_i, w_i \in \mathbb{C}$, $\lambda \in \mathbb{C}$

scaling with a complex number

Proposition 6.12. Properties of the vector space $\mathbb{C}^n$

The set $V = \mathbb{C}^n$ with the addition $+$ and scalar multiplication $\cdot$ fulfils the following:

(1) $\forall \mathbf{v}, \mathbf{w} \in V$: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (+ is commutative)

(2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (+ is associative)

(3) There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ we have $\mathbf{v} + \mathbf{o} = \mathbf{v}$.

(4) For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.

(5) For the number $1 \in \mathbb{C}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.

(6) $\forall \lambda, \mu \in \mathbb{C}$ $\forall \mathbf{v} \in V$: $\lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v}$ ($\cdot$ is associative)

(7) $\forall \lambda \in \mathbb{C}$ $\forall \mathbf{v}, \mathbf{w} \in V$: $\lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w})$ (distributive $\cdot +$)

(8) $\forall \lambda, \mu \in \mathbb{C}$ $\forall \mathbf{v} \in V$: $(\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v})$ (distributive $+\cdot$)

↳ (complex) vector space $\mathbb{C}^n$

↳ same calculation rules as in $\mathbb{R}^n$, just with complex numbers

Definition 6.13. Subspaces in $\mathbb{C}^n$

A nonempty subset $U \subset \mathbb{C}^n$ is called a (linear) subspace of $\mathbb{C}^n$ if all linear combinations of vectors in U remain also in U. This means:

$$\begin{array}{lrcl} (1) & \mathbf{o} \in U\,, & & \\ (2) & \mathbf{u} \in U\,,\ \lambda \in \mathbb{C} & \Longrightarrow & \lambda\mathbf{u} \in U\,, \\ (3) & \mathbf{u}, \mathbf{v} \in U & \Longrightarrow & \mathbf{u} + \mathbf{v} \in U\,. \end{array}$$

Definition 6.14. Span

Let $M \subset \mathbb{C}^n$ be any non-empty subset. Then we define:

$$\mathrm{Span}\,(M) := \{\lambda_1 \mathbf{u}_1 + \cdots + \lambda_k \mathbf{u}_k \;:\; \mathbf{u}_1, \ldots, \mathbf{u}_k \in M\,,\ \lambda_1, \ldots, \lambda_k \in \mathbb{C}\,,\ k \in \mathbb{N}\}\,.$$

This subspace is called the span or the linear hull of M. For convenience, we define $\mathrm{Span}(\varnothing) := \{\mathbf{o}\}$.

$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ with $0 \in \mathbb{C}$

Definition 6.15. Linear dependence and indepedence

A family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ *of* k *vectors from* $\mathbb{C}^n$ *is called* *linearly dependent* *if we find a non-trivial linear combination for* $\mathbf{o}$*. This means that we can find* $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ *that are not all equal zero such that*

$$\sum_{j=1}^{k} \lambda_j \mathbf{v}_j = \mathbf{o}\,.$$

If this is not possible, we call the family $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ *linearly independent. This means that*

$$\sum_{j=1}^{k} \lambda_j \mathbf{v}_j = \mathbf{o} \quad \Rightarrow \quad \lambda_1, \dots, \lambda_k = 0$$

holds.

Definition 6.16. Basis, basis vectors

Let V *be a subspace of* $\mathbb{C}^n$*. A family* $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ *is called a* *basis* *of* V *if*

(a) $V = \mathrm{Span}(\mathcal{B})$ *and*

(b) $\mathcal{B}$ *is linearly independent.*

The elements of $\mathcal{B}$ *are called the* *basis vectors* *of* V*.*

$(e_1, \dots, e_n)$ basis for $\mathbb{C}^n$

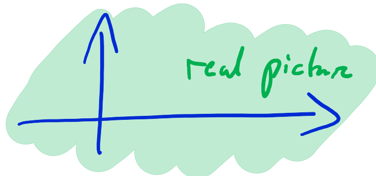
dimension of the subspace

$\dim(V) = k$, $\dim(\mathbb{C}^n) = n$, $\dim(\mathbb{C}) = 1$

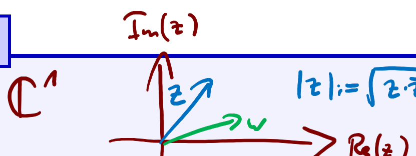
complex dimension

Even in the complex vector space $\mathbb{C}^n$, we are able speak of geometry when endowing the space with an inner product. We try to generalise what we know from the complex plane $\mathbb{C}$ and the real vector space $\mathbb{R}^n$.

$\mathbb{C}$:



Definition 6.17. Inner product in $\mathbb{C}^n$



$|z| := \sqrt{z \cdot \bar{z}} = \sqrt{Re(z)^2 + Im(z)^2}$

$z \cdot \bar{v}$ — measures lengths and angles

For the vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n \quad \text{the number} \quad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 \overline{v_1} + \dots + u_n \overline{v_n} = \sum_{i=1}^{n} u_i \overline{v_i}$$

is called the *(standard) inner product* *of* $\mathbf{u}$ *and* $\mathbf{v}$*. Moreover, we define the real number*

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{|v_1|^2 + \ldots + |v_n|^2}$$

$$= \sqrt{v_1 \cdot \bar{v}_1 + \dots + v_n \bar{v}_n}$$

and call it the *norm* *of* $\mathbf{v}$*.* → measures lengths in $\mathbb{C}^n$

$$\left\| \begin{pmatrix} i \\ 1 \end{pmatrix} \right\| = \sqrt{|i|^2 + |1|^2} = \sqrt{1 + 1} = \sqrt{2}$$

Attention!

In some other books, you might find an alternative definition of the standard inner product in $\mathbb{C}^n$ where the first argument is the complex conjugated one.

Note that $\langle \mathbf{v}, \mathbf{v} \rangle$ is always a real number with ≥ 0 such it gives us indeed a length. Again, we find the important property: $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{o}$.

$\hookrightarrow \|\cdot\| : \mathbb{C}^n \to \mathbb{R} \rightsquigarrow$ (N1), (N2), (N3)

Hence, $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is well-defined and the norm $\|\cdot\|$ has the same properties as in $\mathbb{R}^n$, see Proposition 6.19 below.

Not linear in the second argument!

$$\langle x, \lambda y \rangle \overset{(S4)}{=} \overline{\langle \lambda y, x \rangle} \overset{(S3)}{=} \overline{\lambda \langle y, x \rangle} = \bar{\lambda} \cdot \langle x, y \rangle$$

Proposition 6.18.

The standard inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ fulfils the following: For all vectors $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, one has

- **(S1)** $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ *for* $\mathbf{x} \neq \mathbf{o}$, *(positive definite)*
- **(S2)** $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$, *(additive)* } *(linear)*
- **(S3)** $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, *(homogeneous)* } *(linear)*
- **(S4)** $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$. *(conjugate symmetric)*

Proposition 6.19. Norm

The norm $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ defined by using the standard inner product satisfies for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$:

- **(N1)** $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$, *(positive definite)*
- **(N2)** $\|\alpha \mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$, *(absolutely homogeneous)*
- **(N3)** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. *(triangle inequality).*

Recall: $\langle \cdot, \cdot \rangle$ standard inner product in $\mathbb{R}^n$

$$\left((A^T)_{ik} = (A)_{ki} = a_{ki} \right)$$

$$\langle Ax, y \rangle \overset{\checkmark}{=} \langle x, A^T y \rangle = \sum_i x_i (A^T y)_i$$

$$= \sum_{k=1}^n (Ax)_k y_k = \sum_{k,i} a_{ki} x_i y_k = \sum_{k,i} x_i a_{ki} y_k = \sum x_i (A^T)_{ik} y_k$$

$$\langle Ax, y\rangle = \sum_{k=1}^{n} (Ax)_k \overline{y_k}$$

$$A^T \in \mathbb{R}^{m\times n} \rightsquigarrow A^* \in \mathbb{C}^{m\times n}$$

Proposition & Definition 6.20. Adjoint matrix

For a given matrix

there are other notations

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m\times n}, \quad \text{the matrix } A^* := \overline{A^T} = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n\times m}$$

is called the adjoint matrix *of A. It is the uniquely determined matrix that fulfils the equation*

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^*\mathbf{y}\rangle$$

for all $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$.

Remember: $\langle x, y\rangle \; (= x^T \overline{y}) = y^* x$ (switched rôles)

In analogy to Proposition 6.9 (c), we get the following for complex matrices:

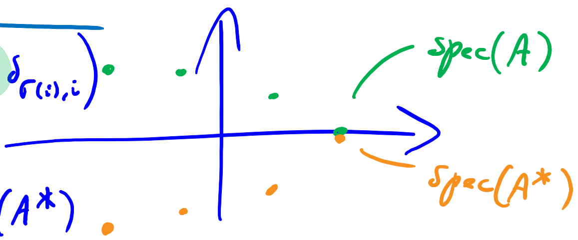
Proposition 6.21. Spectrum of A^*

For all $A \in \mathbb{C}^{n\times n}$, we have $\mathrm{spec}(A^*) = \{\overline{\lambda} : \lambda \in \mathrm{spec}(A)\}$.

Proof: $\det(A^* - \lambda \mathbb{1}) = \overline{\sum_{\sigma\in P_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \left((A^T)_{\sigma(i),i} - \overline{\lambda}\cdot\delta_{\sigma(i),i}\right)}$

$= \overline{\det(A^T - \overline{\lambda}\cdot\mathbb{1})} \Rightarrow \lambda \in \mathrm{spec}(A^*)$

$\Leftrightarrow \overline{\lambda} \in \mathrm{spec}(A^T) = \mathrm{spec}(A)$



Some important notions:

Definition 6.22.

A complex matrix $A \in \mathbb{C}^{n\times n}$ is called

- selfadjoint *if $A = A^*$ (complex version of "symmetric"),*
- skew-adjoint *if $A = -A^*$ (complex version of "skew-symmetric"),*
- unitary *if $AA^* = \mathbb{1} = A^*A$ (complex version of "orthogonal"),*
- normal *if $AA^* = A^*A$.*

Beispiel 6.23. (a) $A = \begin{pmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1} & \overline{-2\mathrm{i}} \\ \overline{2\mathrm{i}} & \overline{0} \end{pmatrix} = \begin{pmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 0 \end{pmatrix} = A$

(b) $A = \begin{pmatrix} \mathrm{i} & -1+2\mathrm{i} \\ 1+2\mathrm{i} & 3\mathrm{i} \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{\mathrm{i}} & \overline{1+2\mathrm{i}} \\ \overline{-1+2\mathrm{i}} & \overline{3\mathrm{i}} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & 1-2\mathrm{i} \\ -1-2\mathrm{i} & -3\mathrm{i} \end{pmatrix} = -A$

(c) $A = \begin{pmatrix} 1+\mathrm{i} & 3-2\mathrm{i} \\ 2\mathrm{i} & -1 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1+\mathrm{i}} & \overline{2\mathrm{i}} \\ \overline{3-2\mathrm{i}} & \overline{-1} \end{pmatrix} = \begin{pmatrix} 1-\mathrm{i} & -2\mathrm{i} \\ 3+2\mathrm{i} & -1 \end{pmatrix} \notin \{A, -A\}$

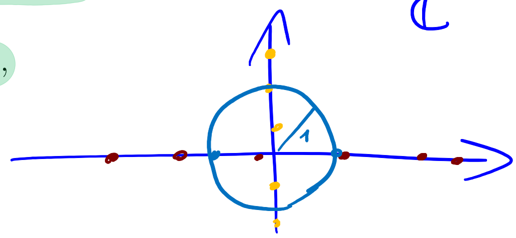
$\hookrightarrow$ normal? No!

$\mathbb{R}^{n\times n} \subset \mathbb{C}^{n\times n}$

If $A \in \mathbb{C}^{n\times n}$ is a real matrix, i.e. $a_{ij} \in \mathbb{R}$ for all i, j, then we get:

$$\begin{aligned} \text{adjoint matrix } A^* &= \text{transpose matrix } A^T, \\ \text{selfadjoint} &= \text{symmetric}, \\ \text{skew-adjoint} &= \text{skew-symmetric}, \\ \text{unitary} &= \text{orthogonal}. \end{aligned}$$

$\mathbb{C}$



Proposition 6.24. Where are the eigenvalues?

(a) If $A^* = A$ (selfadjoint), then all eigenvalues of A lie on the real line.

(b) If $A^* = -A$ (skewadjoint), then all eigenvalues of A lie on the imaginary axis.

(c) If $A^*A = \mathbb{1}$ (unitary), then all eigenvalues of A lie on the unit circle in $\mathbb{C}$.

Proof: Eigenvalue equation $Ax = \lambda x$ with $x \neq 0$ (Normalise $x \rightsquigarrow \|x\| = 1$)

$\Rightarrow \langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda \langle x, x\rangle = \lambda \cdot \|x\|^2 = \lambda$

(a) $\lambda = \langle Ax, x\rangle = \langle x, A^* x\rangle \overset{A \text{ selfadj.}}{=} \langle x, Ax\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$

$= \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$

(b) $\lambda = \langle Ax, x\rangle = \dots = -\overline{\lambda} \Rightarrow \lambda = iy$ with $y \in \mathbb{R}$

(c) $\lambda \cdot \overline{\lambda}\langle x, y\rangle = \langle \lambda x, \lambda y\rangle = \langle Ax, Ay\rangle = \langle x, A^*A y\rangle = 1 \langle x, y\rangle$

$\overset{y=x}{\Rightarrow} \lambda \cdot \overline{\lambda} = 1 \Rightarrow |\lambda| = 1 \rightarrow \lambda$ lies on unit circle

6.4 Eigenvalues and similarity

Homework: f_A injective $\Leftrightarrow$ f_B injective

Definition 6.25. Similarity

Two matrices $A, B \in \mathbb{C}^{n\times n}$ are called <u>similar</u> if there is an invertible matrix $S \in \mathbb{C}^{n\times n}$ with $A = SBS^{-1}$.

$\Rightarrow$ Later: change of basis

Proposition 6.26.

Similar matrices have the same characteristic polynomial and thus the same eigenvalues.

Proof: $\underbrace{\det(A - \lambda \mathbb{1})}_{p_A(\lambda)} = \det(SBS^{-1} - \lambda \mathbb{1}) = \det(S(B - \lambda\mathbb{1})S^{-1})$

$= \det(S) \cdot \det(B - \lambda \mathbb{1}) \cdot \det(S^{-1}) = \det(B - \lambda \mathbb{1}) = p_B(\lambda)$

$$A = S^{-1}\begin{pmatrix} \lambda_1 & & (*) \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S \implies \lambda_1, \dots, \lambda_n \text{ eigenvalues of } A.$$

Remark:

Later, we will see that any matrix $A \in \mathbb{C}^{n\times n}$ is similar to a triangular matrix.

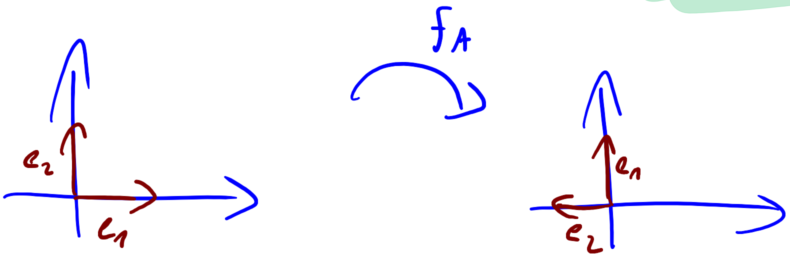
6.5 Calculating eigenvectors

$\mathbb{R}^{n\times n} \subset \mathbb{C}^{n\times n}$

Even for matrices $A \in \mathbb{R}^{n\times n}$, we now consider the eigenvalues in $\mathbb{C}$ and the eigenvectors in $\mathbb{C}^n$. This means that we now consider all square matrices as matrices in $\mathbb{C}^{n\times n}$.

Example 6.27. Consider $A \in \mathbb{R}^{2\times 2}$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $p_A(\lambda) = \lambda^2 + 1$ and $\mathrm{spec}(A) = \{-\mathrm{i}, \mathrm{i}\}$.

The corresponding map $f_A : \mathbb{R}^2 \to \mathbb{R}^2$ rotates $\mathbf{e}_1$ and $\mathbf{e}_2$, and hence any vector in $\mathbb{R}^2$, by an angle of $\frac{\pi}{2}$ (or 90°) in positive sense. In this sense, no line is sent to itself again. However, this is only a problem if we look at the "real" picture.



Proposition 6.28. Spectrum is not empty

For a square matrix $A \in \mathbb{C}^{n\times n}$ holds:

(a) $\mathrm{spec}(A) \neq \emptyset$.

(b) A is invertible if and only if $0 \notin \mathrm{spec}(A)$.

Proof: (a) Fundamental theorem of algebra

(b) A is invertible $\iff 0 = \det(A) = \det(A - 0\mathbb{1}) \iff 0 \in \mathrm{spec}(A)$

Looking at Proposition 6.4, we see what we have to do in order to calculate the eigenvectors of a given matrix A if we already know the eigenvalues λ:

Definition 6.29. Eigenspace

The solution set of the LES $(A - \lambda\mathbb{1})\mathbf{x} = \mathbf{o}$, *which means* $\operatorname{Ker}(A - \lambda\mathbb{1})$, *is called the* eigenspace *with respect to the eigenvalue* λ *and denoted by* $\operatorname{Eig}(\lambda)$. *Each* ***nonzero*** *vector* $\mathbf{x} \in \operatorname{Eig}(\lambda) \setminus \{\mathbf{o}\}$ *is an eigenvector w.r.t. the eigenvalue* λ.

For all $\lambda \in \mathbb{C}$: $\operatorname{Eig}(\lambda)$ is linear subspace

$\operatorname{Eig}(\lambda) = \{o\} \iff \lambda$ is not an eigenvalue

Example 6.30. Consider $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$: $\mathbf{x}_i \neq \mathbf{o}$ is an eigenvalue for λ_i with $i \in \{1, 2\}$ if

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, \qquad \text{i.e.} \qquad (A - \lambda_i\mathbb{1})\mathbf{x}_i = \mathbf{o}.$$

Hence, we have to solve the LES $(A - \lambda_1\mathbb{1})\mathbf{x}_1 = \mathbf{o}$ and $(A - \lambda_2\mathbb{1})\mathbf{x}_2 = \mathbf{o}$.

$$\underline{\lambda_1 = 4}: \qquad A - \lambda_1\mathbb{1} = \begin{pmatrix} 3 - \lambda_1 & 2 \\ 1 & 2 - \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix},$$

$$(A - \lambda_1\mathbb{1})\mathbf{x}_1 = \mathbf{o}$$

In the same manner:

$$\underline{\lambda_2 = 1}: \qquad A - \lambda_2\mathbb{1} = \begin{pmatrix} 3 - \lambda_2 & 2 \\ 1 & 2 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix},$$

$$(A - \lambda_2\mathbb{1})\mathbf{x}_2 = \mathbf{o}$$

Definition 6.31. Multiplicities

Let $A \in \mathbb{C}^{n \times n}$ *be square matrix. Then the characteristic polynomial can be written as:*

$$p_A(z) = (\lambda_1 - z)^{\alpha_1} \cdot (\lambda_2 - z)^{\alpha_2} \cdots (\lambda_k - z)^{\alpha_k} \tag{6.2}$$

where $\lambda_1, \ldots, \lambda_k$ *are pairwise different. The natural number* α_j *above is called:*

$$\alpha(\lambda_j) := \alpha_j \qquad \text{algebraic multiplicity of } \lambda_j$$

and tells you how often the eigenvalue λ_j *occurs in the characteristic polynomial. We also define*

$$\gamma(\lambda_j) := \dim \operatorname{Eig}(\lambda_j) = \dim(\operatorname{Ker}(A - \lambda_j\mathbb{1})) \qquad \text{geometric multiplicity of } \lambda_j$$