German: Eigenwert (Hilbert 1904)
(proper value, own value
characteristic value)

6 Eigenvalues and similar things

Consider again a square matrix $A \in \mathbb{R}^{n\times n}$ and the associated linear map $f_A : \mathbb{R}^n \to \mathbb{R}^n$ which maps $\mathbb{R}^n$ into itself.

same direction

Question:

Are there vectors $\mathbf{v}$ *which are only scaled by* f_A*? This means that they satisfy:*

$$A\mathbf{v} = \lambda\mathbf{v} \quad \text{or equivalently} \quad (A - \lambda\mathbb{1})\mathbf{v} = \mathbf{o}$$

Kernel of $A - \lambda\mathbb{1}$

- λ *is called* eigenvalue *of* A,
- $\mathbf{v}$ *is called* eigenvector *of* A *(if* $\mathbf{v} \neq \mathbf{o}$*).*

First conclusions:

- Not very interesting (trivial): $\mathbf{v} = \mathbf{o}$.
- $\mathbf{v} \in \mathrm{Ker}(A) \setminus \{\mathbf{o}\} \Rightarrow A\mathbf{v} = 0\mathbf{v}$, so $\lambda = 0$.
- $\mathbf{v} \in \mathrm{Ker}(A - \lambda\mathbb{1}) \setminus \{\mathbf{o}\} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}$, so λ is an eigenvalue.
- $\mathbf{v}$ eigenvector $\Rightarrow \alpha\mathbf{v}$ is also an eigenvector (for $\alpha \neq 0$).

Example: (a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $Av = \lambda v$ $\overset{\text{For } v \neq 0}{\Longleftrightarrow}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Longleftrightarrow \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ E_2

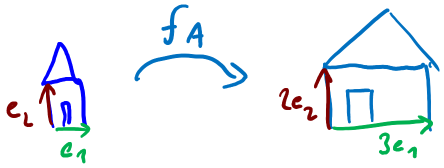
(3 unknowns, 2 equations, not a linear system)

$\overset{E_2}{\Longrightarrow}$ $\lambda = 1$ or $v_2 = 0$ $\overset{E_1}{\Longrightarrow}$ $\lambda = 1$ or $v_1 = 0$ $\Rightarrow v = o$ (excluded!)

$\overset{E_1}{\Longrightarrow}$ $\lambda = 1$ and $v_2 = 0$ $\Longrightarrow$ $\lambda = 1$, $v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ for all $v_1 \in \mathbb{R}\setminus\{0\}$

(b) $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $Bv = \lambda v$ $\Longleftrightarrow$ $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

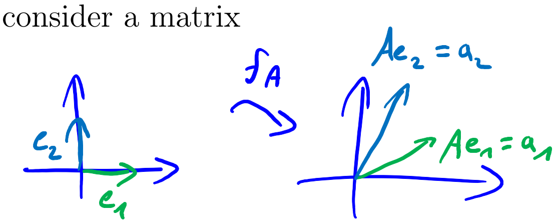
$\Longleftrightarrow \begin{pmatrix} 3v_1 \\ 2v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\Longrightarrow$ $\lambda = 2$, $v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$ or $\lambda = 3$, $v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$



6.1 What is an eigenvalue and an eigenvector?

We start with an illustration in two-dimensional cases and consider a matrix

$$A = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} \in \mathbb{R}^{2\times 2}$$



and the associated linear map $f_A : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{x} \mapsto A\mathbf{x}$.

$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

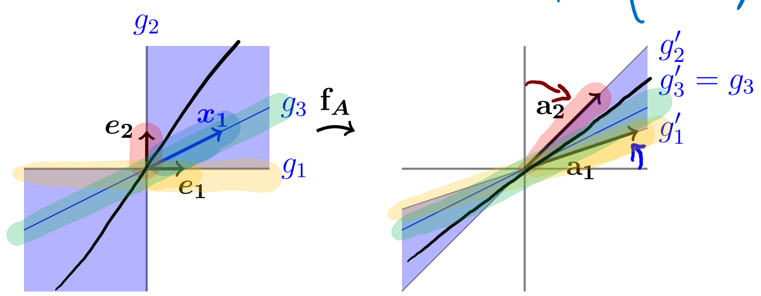
Images of the lines:

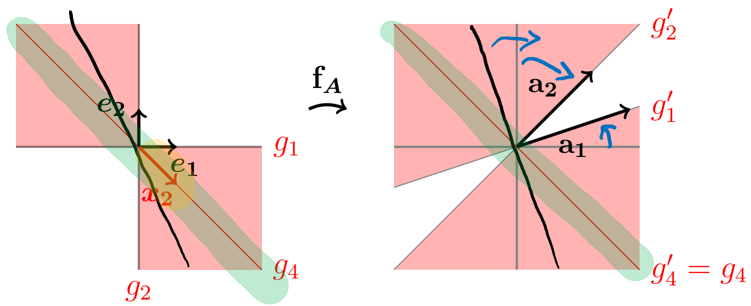
$f_A(g_1) = g_1'$

$f_A(g_2) = g_2'$

Stable line: $f_A(g_3) = g_3$

(Eigenvector direction!)





In the same sense, we can look at the other quadrants of our coordinate system. There we also find such a special line:

$$g_4' := f_A(g_4) = g_4.$$

For points $0 \neq x_1 \in g_3$, $0 \neq x_2 \in g_4$, $Ax_1 \in g_3$, $Ax_2 \in g_4$

$\begin{pmatrix} g_3 = \text{span}(x_1) \\ g_4 = \text{span}(x_2) \end{pmatrix}$

$\Rightarrow Ax_1 = \lambda_1 \cdot x_1$ for some $\lambda_1 \in \mathbb{R}$

$\Rightarrow Ax_2 = \lambda_2 \cdot x_2$ for some $\lambda_2 \in \mathbb{R}$

Definition 6.1. Eigenvalue, Eigenvector, spectrum

Let A be a square matrix. A vector $\mathbf{x} \neq \mathbf{o}$ is called an eigenvector of A, if $A\mathbf{x}$ is a multiple of $\mathbf{x}$. This scalar λ, which means $A\mathbf{x} = \lambda\mathbf{x}$, is called eigenvalue of A. The set of all eigenvalues of A is called the spectrum of A and denoted by spec(A).

$\rightarrow$ Later: eigenvalues $\subset \mathbb{C}$

$\rightarrow$ matrices $A \in \mathbb{C}^{n\times n}$

In our example above: $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6+2 \\ 2+2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \lambda = 4 \text{ eigenvalue}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{eigenvector}$$

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1-2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \lambda = 1 \text{ eigenvalue}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{eigenvector}$$

This is very general definition and will work later for other cases in the same manner. Here, we are first interested in matrices $A \in \mathbb{R}^{n \times n}$ and eigenvalues $\lambda \in \mathbb{R}$. However, you may already see that this can also work for complex numbers. We may also include $\lambda \in \mathbb{C}$ later.

Proposition 6.2. Multiple of eigenvector = eigenvector

Every multiple (not $\mathbf{o}$*) of an eigenvector* $\mathbf{x}$ *for* A *is also an eigenvector for* A*, corresponding to the same eigenvalue* λ*.*

Proof: $Ax = \lambda x \iff \alpha(Ax) = \alpha(\lambda x), \quad \alpha \neq 0$

$\iff A(\alpha \cdot x) = \lambda(\alpha \cdot x)$

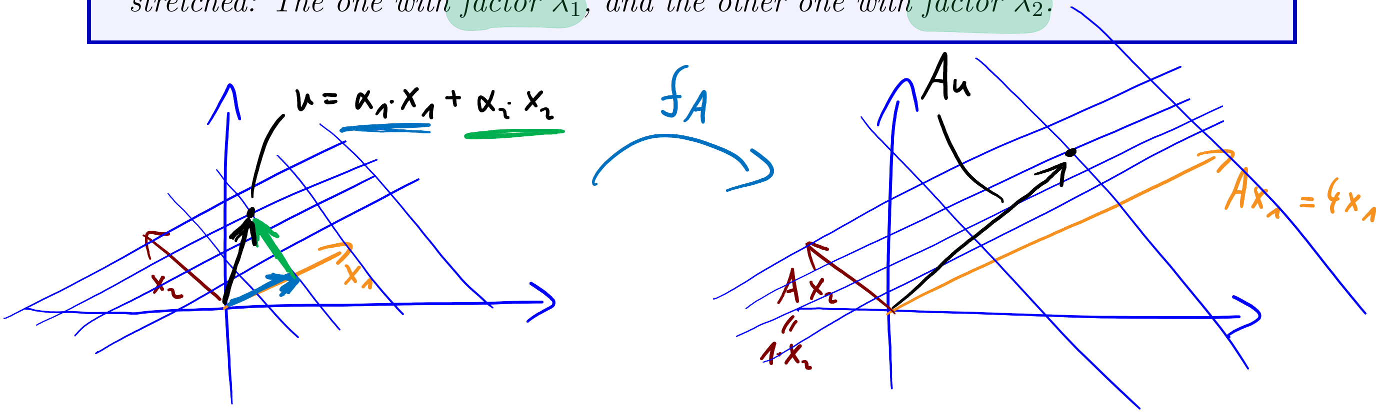
Looking again at the pictures above:

- We have $A\mathbf{x} = \lambda_1 \mathbf{x}$ for all multiples $\mathbf{x}$ of $\mathbf{x}_1 \in g_3$ (which means for all $\mathbf{x} \in g_3$).
- Also we have $A\mathbf{x} = \lambda_2 \mathbf{x}$ for all multiples $\mathbf{x}$ of $\mathbf{x}_2 \in g_4$ (which means for all $\mathbf{x} \in g_4$).
- Looking at the line g_3, the map f_A acts like scaling with the factor λ_1.
- Looking at the line g_4, the map f_A acts like scaling with the factor λ_2.

Optimal

Perfect coordinate system for the map f_A

Describing $\mathbb{R}^2$ *with a coordinate system given by the two lines* g_3 *and* g_4 *(instead of* g_1 *and* g_2*), the acting of the map* f_A *is very simple: The coordinate axes are only stretched: The one with factor* λ_1*, and the other one with factor* λ_2*.*



$Au = \alpha_1 (Ax_1) + \alpha_2 (Ax_2)$

$= \alpha_1 (4 x_1) + \alpha_2 (1 \cdot x_2)$

To get this "perfect coordinate system" we need all the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Question:

(a) How to find the eigenvalues and the eigenvectors of A?

(b) Do you always find n eigenvalues for an $n \times n$ matrix A?

(c) Do you find n different directions for eigenvectors?

(d) How to change the coordinate system?

(e) What are applications for this?

6.2 The characteristic polynomial

Our goal is to find $\lambda \in \mathbb{R}$ and $\mathbf{x} \neq \mathbf{o}$ such that $(A - \lambda \mathbb{1})\mathbf{x} = \mathbf{o}$, i.e., $(A - \lambda \mathbb{1})$ has a nontrivial kernel. This means that the corresponding map for $A - \lambda \mathbb{1}$ is not injective and, hence, it is a singular matrix.

Idea:

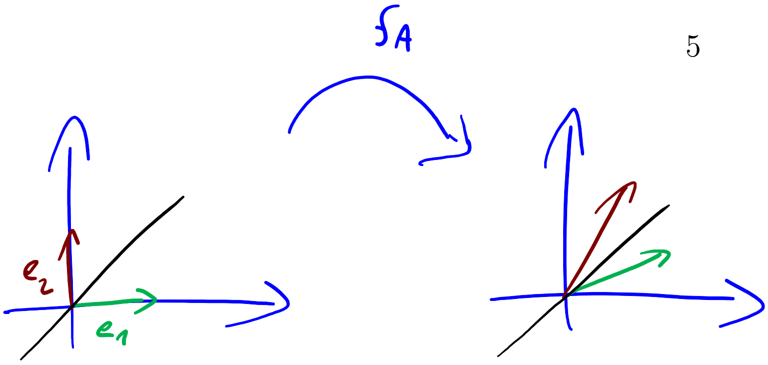
Compute $\det(A - \lambda \mathbb{1})$, *which yields a* polynomial *of degree n in λ and determine its zeros, because*

$$\begin{aligned} \underline{\det(A - \lambda \mathbb{1}) = 0} \quad &\Leftrightarrow \quad A - \lambda \mathbb{1} \text{ is singular} \\ &\Leftrightarrow \quad \operatorname{Ker}(A - \lambda \mathbb{1}) \text{ is non-trivial} \\ &\Leftrightarrow \quad \underline{\lambda} \text{ is an eigenvalue} \end{aligned}$$

Then, compute a basis for $\operatorname{Ker}(A - \lambda \mathbb{1})$ *for each eigenvalue.*

solving an LES

$\underset{\neq 0}{x} \in \operatorname{Ker}(A - \lambda \mathbb{1})$ eigenvector



Example 6.3.

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

$$\det(A - \lambda\mathbb{1}) = \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda) - 2\cdot 1 = 10 - 7\lambda + \lambda^2 = 0$$

quadratic equation

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7\pm 3}{2} \Rightarrow \lambda_1 = 2, \lambda_2 = 5$$

Thus we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Let us compute the eigenvectors:

$$\mathbf{o} = (A - 2\mathbb{1})\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + 2v_2 \\ v_1 + 2v_2 \end{pmatrix} \Rightarrow \mathbf{v} = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad , \alpha \in \mathbb{R}$$

$$\mathbf{o} = (A - 5\mathbb{1})\mathbf{v} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + 2v_2 \\ v_1 - v_2 \end{pmatrix} \Rightarrow \mathbf{v} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \alpha \in \mathbb{R}$$

$\rightsquigarrow$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector of A, to eigenvalue $\lambda_1 = 2$ $\qquad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector of A to eigenvalue of $\lambda_2 = 5$

Proposition 6.4. Five properties of an eigenvalue

For a square matrix A and a number λ the following is equivalent:

(i) λ is an eigenvalue A.

(ii) There is a vector $\mathbf{x} \neq \mathbf{o}$ with $A\mathbf{x} = \lambda\mathbf{x}$.

(iii) The space $\mathrm{Ker}(A - \lambda\mathbb{1})$ contains a vector $\mathbf{x} \neq \mathbf{o}$.

(iv) The matrix $A - \lambda\mathbb{1}$ is not invertible.

(v) $\det(A - \lambda\mathbb{1}) = 0$

$$\det(A - \lambda\mathbb{1}) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & \vdots \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ \vdots & \cdots & & a_{nn}-\lambda \end{pmatrix}$$

Let $A \in \mathbb{R}^{n\times n}$. Then we observe that $\det(A - \lambda\mathbb{1}) = p_A(\lambda)$ is a polynomial of order n in the variable λ. For example, there could be coefficients c_i such that

$$p_A(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots c_1\lambda + c_0\,.$$

Definition 6.5. Characteristic polynomial

For an $n \times n$-Matrix A, the polynomial $\lambda \mapsto \det(A - \lambda \mathbb{1})$ is called the characteristic polynomial of the matrix A and is denoted by p_A.

other notations are possible

Example 6.6. Look at $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

$$p_A(\lambda) = \det(A - \lambda \mathbb{1}) = \det\left(\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}$$
$$= (3-\lambda)\cdot(2-\lambda) - 2\cdot 1 = 6 - 3\lambda - 2\lambda + \lambda^2 - 2 = \underbrace{\lambda^2 - 5\lambda + 4}_{\text{char. polynomial for } A.}$$

Solving the quadratic equation:

$$\lambda_{1,2} = -\frac{-5}{2} \pm \sqrt{\frac{25}{4} - 4} = \frac{5}{2} \pm \sqrt{\frac{9}{4}} = \frac{5 \pm 3}{2} \in \{1, 4\}, \quad \text{hence} \quad \lambda_1 = 4, \lambda_2 = 1.$$

$$\operatorname{Ker}(A - \overset{\lambda_2}{1}\cdot\mathbb{1}) = \operatorname{Ker}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = ?$$
$$\operatorname{Ker}(A - 4\cdot\mathbb{1}) = \operatorname{Ker}\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} = ?$$

Theorem 6.7. Fundamental theorem of algebra (Gauß 1799)

Let $a_0, a_1, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$. Then the polynomial equation

$$\underbrace{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0}_{=:p(x)} = 0$$

*has n (not necessarily different) solutions $x_1, \ldots, x_n$ **in** $\mathbb{C}$. Moreover, we find for $x \in \mathbb{C}$:*

$$p(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n).$$

need $\mathbb{C}$

Looking at p_A:

$\rightarrow$ $p_A(\lambda) = 0$ has at least one solution in $\mathbb{C}$

$\Rightarrow$ A has at least one eigenvalue in $\mathbb{C}$

$\left(p_A(\lambda) = \lambda^2 + 1 \Rightarrow \lambda_{1,2} = \pm i \right)$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\rightarrow$ $p_A(\lambda) = 0$ multiple roots $\Rightarrow$ multiplicity of an eigenvalue

$p_A(x) = (\pm 1)(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

for example: $p_A(x) = (x-1)^2(x-3)^4$

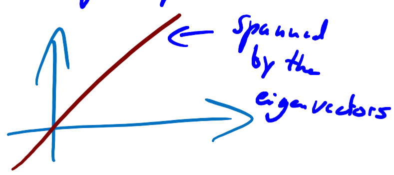
Definition 6.8.

If the same eigenvalue λ appears $\alpha(\lambda)$ times in this factorisation, we say:

$$\lambda \text{ has } \textit{algebraic multiplicity } \alpha(\lambda) \ .$$

- If we have k *different* eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ ($k \leq n$), then $\alpha(\lambda_1) + \cdots + \alpha(\lambda_k) = n$, because polynomials of degree n can be factorised into n linear factors.
- If λ is an eigenvalue, then $A - \lambda\mathbb{1}$ is singular, so $\gamma(\lambda) := \dim(\operatorname{Ker}(A - \lambda\mathbb{1})) \geq 1$.

↳ geometric multiplicity



Proposition 6.9. Spectrum for triangular matrices

Let $A \in \mathbb{R}^{n\times n}$ a square matrix.

(a) For a matrix in triangular form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix},$$

we get $\operatorname{spec}(A) = \{a_{11}, a_{22}, \ldots, a_{nn}\}$.

(b) For a square block matrix in triangular form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

with square matrices B and D, we get $\operatorname{spec}(A) = \operatorname{spec}(B) \cup \operatorname{spec}(D)$.

(c) Also $\operatorname{spec}(A) = \operatorname{spec}(A^T)$. *Hence (a) and (b) also hold for lower triangular matrices.*

Proof. (c) λ eigenvalue of A $\iff \det(A - \lambda\mathbb{1}) = 0$

$\iff \det((A-\lambda\mathbb{1})^T) = 0$

$\iff \det(A^T - \lambda\mathbb{1}) = 0$

$\iff \lambda$ eigenvalue of A^T

(b), (a)

$\square$