$n=2 \rightsquigarrow 2!$ terms

$n=3 \rightsquigarrow 6$ terms

$n=4 \rightsquigarrow 24$ terms

↳ $\mathbb{R}^n \rightsquigarrow$ Leibniz formula ($n!$ terms)

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix} \qquad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

VL10 ↓

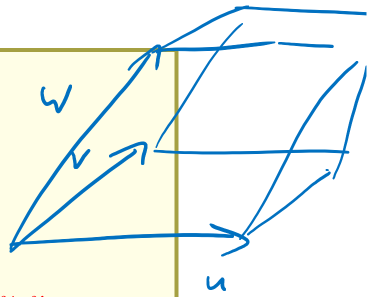
Rule of thumb: Rule of Sarrus (Only for $n = 3$)

$$\begin{matrix} \boldsymbol{u_1} & \boldsymbol{v_1} & \boldsymbol{w_1} & u_1 & v_1 \\ \boldsymbol{u_2} & \boldsymbol{v_2} & \boldsymbol{w_2} & u_2 & v_2 \\ \boldsymbol{u_3} & \boldsymbol{v_3} & \boldsymbol{w_3} & u_3 & v_3 \end{matrix} \qquad - \qquad +$$

$\det\begin{pmatrix} \vdots & \vdots & \vdots \\ u & v & w \\ \vdots & \vdots & \vdots \end{pmatrix}$

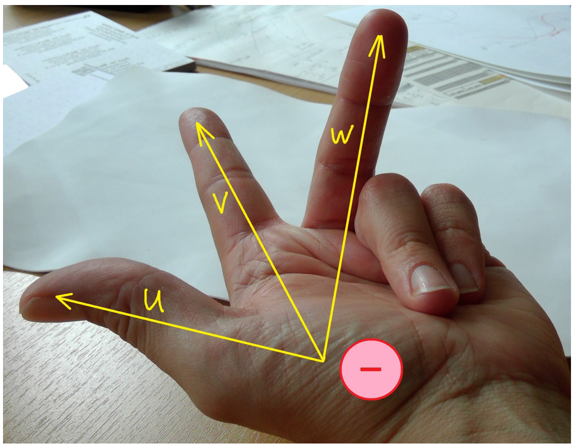
$\|$

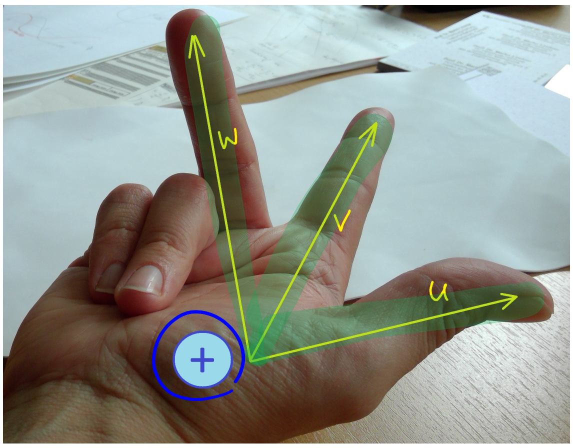
$$\mathrm{Vol}_n(\mathbf{u}, \mathbf{v}, \mathbf{w}) = +u_1v_2w_3 + v_1w_2u_3 + w_1u_2v_3 - u_3v_2w_1 - v_3w_2u_1 - w_3u_2v_1$$



Don't do it for $\mathbb{R}^{4\times 4}$

Moreover, the sign of the three-dimensional volume can be easily seen by the right-hand-rule:





4.3 The cofactor expansion

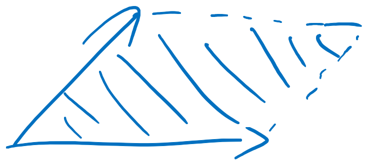
We already know that volume measure and the determinant of a matrix coincide. From now on, we will only consider the determinant of matrices and keep in mind that this is the volume spanned by the columns of the matrix. We consider a matrix

$$A = (a_{ij}) \quad \text{with} \quad i, j = 1, \ldots, n.$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

We have already encountered these cases:

- $A \in \mathbb{R}^{1\times 1}$:
$$\det(A) = a_{11}$$
- $A \in \mathbb{R}^{2\times 2}$:
$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$
- $A \in \mathbb{R}^{3\times 3}$:
$$\begin{aligned}\det(A) = &+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$



Note that it would be helpful to have an algorithm that reduces an $n \times n$-matrix to these cases.

Checkerboard of signs:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The entry (i, j) of this matrix is $(-1)^{i+j}$.

Cofactors:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$(n-1) \times (n-1)$

For $A \in \mathbb{R}^{n\times n}$ pick one entry a_{ij}.

- Delete the i^{th} row and the j^{th} column of A.
- Call the remaining matrix $A^{(i,j)} \in \mathbb{R}^{(n-1)\times(n-1)}$.
- The cofactor c_{ij} of a_{ij} is defined as

$$c_{ij} := (-1)^{i+j} \det(A^{(i,j)})$$

checkerboard

Example:

$$A = \begin{pmatrix} 1^+ & 2^- & 3^+ \\ 4^- & 5^+ & 6^- \\ 7^+ & 8^- & 9^+ \end{pmatrix} \rightsquigarrow A^{(2,1)} = \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix}$$

$$C_{21} = (-1) \cdot \det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} = (-1) \cdot (18 - 24) = \underline{6}$$

Definition 4.10.

$C = (c_{ij}) = (-1)^{i+j}\det(A^{(i,j)})$ *is called the* *cofactor matrix* *of* A.

$A \in \mathbb{R}^{n\times n} \rightsquigarrow C \in \mathbb{R}^{n\times n}$

Example: $A \in \mathbb{R}^{2\times 2}$, $A = \begin{pmatrix} a^{+} & b^{-} \\ c^{-} & d^{+} \end{pmatrix}$, $c_{11} = +d$, $c_{12} = -c$, $c_{22} = a$, $c_{21} = -b$

$\rightsquigarrow C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Proposition 4.11. Laplace's formula

Let $A \in \mathbb{R}^{n\times n}$ *and choose the* j^{th} *column:*

$$\det(A) = \sum_{i=1}^{n} a_{ij}c_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A^{(i,j)})$$

checkerboard

j^{th} column

This is *expanding* $\det(A)$ *along the* j^{th} *column.*

If you choose the i^{th} *row:*

i^{th} row

$$\det(A) = \sum_{j=1}^{n} a_{ij}c_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A^{(i,j)})$$

One calls this *expanding* $\det(A)$ *along the* i^{th} *row.*

Proof with Leibniz formula

To compute $\det(A^{(i,j)})$, apply the same formula recursively, until you reach 2×2 matrices, where the corresponding formula can be applied. The proof of this formula follows immediately from the Leibniz formula and is left to the reader.

Rule of thumb: Do not forget the checkerboard matrix

Remember the signs when expanding a matrix along a column or a row.

$$\begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{pmatrix}$$

Rule of thumb: Use the nothingness.

Since it is your choice which of the rows or the columns you want to expand along, you should search for zeros. If you find a row or column with a lot of zeros, which means that one has $a_{ij} = 0$ *for some indices, you do not need to calculate* $\det(A_{ij})$ *for these indices.*

Example 4.12. Consider the matrix

$$A = \begin{pmatrix}0&2&3&4\\2&0&0&0\\1&1&0&0\\6&0&1&2\end{pmatrix}. \quad \in \mathbb{R}^{4\times 4} \longrightarrow \text{expand along this row}$$

Here, it would be useful first to expand along the second row since we find three zeros there:

$$\det(A) = \det\begin{pmatrix}0&2&3&4\\2&0&0&0\\1&1&0&0\\6&0&1&2\end{pmatrix} = (-2)\cdot\det\begin{pmatrix}2&3&4\\1&0&0\\0&1&2\end{pmatrix} = (-2)\cdot(-1)\cdot\det\begin{pmatrix}3&4\\1&2\end{pmatrix} = \underline{4}\,.$$

expand along this row

$$\begin{pmatrix}| & & |\\ a_1 & \cdots & a_n\\ | & & |\end{pmatrix}$$

If C is the cofactor matrix of $A \in \mathbb{R}^{n\times n}$, then each entry of C is given by

$\in \mathbb{R}^{n\times n}$ ✓

$$c_{ij} = (-1)^{i+j}\det(A^{(i,j)}) = \det\begin{pmatrix}| & & | & | & | & & |\\ \mathbf{a}_1 & \cdots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n\\ | & & | & | & | & & |\end{pmatrix}$$

i^{th} row j^{th} column

canonical unit vector

$$= \det\begin{pmatrix}| & & | & 0 & | & & |\\ & & & \vdots & & & \\ & & & 0 & & & \\ a_1 & \cdots & a_{j-1} & 1 & a_{j+1} & \cdots & a_n\\ & & & 0 & & & \\ & & & \vdots & & & \\ | & & | & 0 & | & & |\end{pmatrix}$$

Proposition 4.13.

Let $A \in \mathbb{R}^{n\times n}$ and C be its cofactor matrix. Then

$$C^T A = \det(A)\mathbb{1}_n\,. = \begin{pmatrix}\det(A) & & \\ & \ddots & \\ & & \det(A)\end{pmatrix}$$

In particular, if $\det(A) \neq 0$, then A is invertible and the inverse is given by

$$A^{-1} = \frac{C^T}{\det(A)}\,.$$

Proof. This is just a matrix multiplication where we consider the $(i,j)^{\text{th}}$ entry:

i^{th} row
j^{th} column

$$(C^T A)_{ij} = \sum_{k=1}^n c_{ki}a_{kj} = \sum_{k=1}^n \det\begin{pmatrix}| & & | & | & | & & |\\ \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{e}_k & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n\\ | & & | & | & | & & |\end{pmatrix} a_{kj}$$

$$\parallel$$
$$\sum_{k=1}^n (C^T)_{ik}(A)_{kj}$$

$$= \det\begin{pmatrix}| & & | & | & | & & |\\ \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{a}_j & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n\\ | & & | & | & | & & |\end{pmatrix}$$

$= \sum a_{kj} e_k$

det (volume form) is linear in each column

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \cdots \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

This is $\det(A)$ for $i = j$ and otherwise just zero. □

Example:

$A \in \mathbb{R}^{2\times 2}$: $\quad A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \quad C = \begin{pmatrix}d & -c\\ -b & a\end{pmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{a d - b \cdot c} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Remember

4.4 Important facts and using Gauß

Proposition 4.14.

If the square matrix $A \in \mathbb{R}^{n\times n}$ is in triangular form,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{pmatrix},$$

then the determinant is given by the multiplication of the diagonal:

$$\det(A) = a_{11}a_{22}\cdots a_{nn}\,.$$

Proof. Use Laplace's formula to the first column recursively. □

Proposition 4.15. Determinants for block matrices

Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{k\times k}$ two square matrices. For every matrix $B \in \mathbb{R}^{n\times k}$ define a so-called block matrix in triangular form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{R}^{(n+k)\times(n+k)},$$

$$\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots & \vdots & & \\ a_{n1} & \cdots & a_{nn} & b_{n1} & \cdots & b_{nk} \\ \hline 0 & \cdots & 0 & C_{11} & \cdots & C_{1k} \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & C_{k1} & \cdots & C_{kk} \end{array}\right)$$

where 0 denotes the $(k \times n)$ zero matrix. Then, one has:

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C).$$

(Works only in triangular form!)

⇝ Prove it with Leibniz

Proposition 4.16.

For each matrix $A \in \mathbb{R}^{n\times n}$, we have $\det(A^T) = \det(A)$.

Proof. We can use the definition by the Leibniz formula:

$$\det(A) = \sum_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i} = \sum_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma)\; a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n}\,.$$

For the transpose A^T we find the following:

$$\det(A^T) = \sum_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma\in\mathcal{P}_n} \operatorname{sgn}(\sigma)\; a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}\,.$$

We only have to show that both sums consist of the same summands.

By definition of the signum function:

$$\operatorname{sgn}(\sigma\circ\omega) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\omega)$$

From this we get:

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$$

The multiplication is commutative and we can rearrange the product $a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n}$. Hence, we get

$$\operatorname{sgn}(\sigma)\ a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n} = \operatorname{sgn}(\sigma^{-1})\ a_{1,\sigma^{-1}(1)}a_{2,\sigma^{-1}(2)}\cdots a_{n,\sigma^{-1}(n)},$$

Now we substitute ω for σ^{-1} and recognise that all summands are, in fact, the same. Here, it is important that $\mathcal{P}_n$ is a so-called *group*, in which each element has exactly one inverse. Therefore, we can sum over ω instead of σ without changing anything. In summary, we have:

$$\sum_{\sigma\in\mathcal{P}_n}\operatorname{sgn}(\sigma)\ a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n} = \sum_{\omega\in\mathcal{P}_n}\operatorname{sgn}(\omega)\ a_{1,\omega(1)}a_{2,\omega(2)}\cdots a_{n,\omega(n)}\ . \qquad \square$$

Important

det(A+B) ≠ det(A) + det(B) in general

Proposition 4.17.

For $A, B \in \mathbb{R}^{n\times n}$, we have

$$\det(AB) = \det(A)\det(B).$$

multiplicative

In particular, if A is invertible, we have $\det(AA^{-1}) = \det(\mathbb{1}) = 1$

$$\det(A^{-1}) = \frac{1}{\det(A)} \qquad \textit{and} \qquad \det(A^{-1}BA) = \det(B)$$

Fact: det(A) = 0 <=> A is not invertible

Proof. Denote the column vectors of A by $\mathbf{a}_j$ and the rows of B by $\boldsymbol{\beta}_j^T$, we can write the matrix product as $AB = \sum_j \mathbf{a}_j\boldsymbol{\beta}_j^T$ and get:

$B = \begin{pmatrix} -\beta_1^T- \\ \vdots \\ -\beta_n^T- \end{pmatrix}$

$AB = \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}\begin{pmatrix} -\beta_1^T- \end{pmatrix} + \cdots + \begin{pmatrix} | \\ a_n \\ | \end{pmatrix}\begin{pmatrix} -\beta_n^T- \end{pmatrix} = \begin{pmatrix} | & | & | & | \end{pmatrix}$ $\left(\beta_j^T = (b_{j1} \cdots b_{jn}) \right)$

first entry

$$\det(AB) = \operatorname{Vol}_n\left(\sum_{j_1}\mathbf{a}_{j_1}b_{j_1,1},\ldots,\sum_{j_n}\mathbf{a}_{j_n}b_{j_n,n}\right).$$

Now we can use the properties (1), (2), (3), and (4) the volume form has, see Definition 4.4. We get:

Permutation: σ is a bijective map σ: {1,...,n} → {1,...,n} "j₁ = σ(1)"

$$\begin{aligned}\det(AB) &= \sum_{j_1,\ldots,j_n} b_{j_1,1}\cdots b_{j_n,n}\operatorname{Vol}_n(\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_n}) \\ &= \sum_{\sigma\in\mathcal{P}_n} b_{\sigma(1),1}\cdots b_{\sigma(n),n}\operatorname{Vol}_n(\mathbf{a}_{\sigma(1)},\ldots,\mathbf{a}_{\sigma(n)}) \\ &= \sum_{\sigma\in\mathcal{P}_n} b_{\sigma(1),1}\cdots b_{\sigma(n),n}\operatorname{sgn}(\sigma)\operatorname{Vol}_n(\mathbf{a}_1,\ldots,\mathbf{a}_n) \\ &= \det(A)\sum_{\sigma\in\mathcal{P}_n}\operatorname{sgn}(\sigma)b_{\sigma(1),1}\cdots b_{\sigma(n),n} = \det(A)\det(B).\end{aligned} \qquad \square$$

The determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is therefore *multiplicative.* This is what we can use for calculating determinants with the Gaussian elimination since this is nothing more than multiplying matrices from the left.

$A \rightsquigarrow Z_{i+\alpha j} A$, $Z_{i+\alpha j} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ \alpha & & \ddots & \\ & & & 1 \end{pmatrix}$ triangular form

rows

$i \neq j$ ith column $\det(Z_{i+\alpha j}) = 1$

Corollary 4.18.

Using the row operations $Z_{i+\alpha j}$ do not change the determinant. The switching of rows only changes the sign of the determinant. However, scaling rows with a diagonal matrix D changes the determinant by the product of these scaling factors.

$i \neq j$

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \rightsquigarrow \det(D) = d_1 \cdots d_n$$

, $\det(A) = \det(A^T)$

Rule of thumb: Using Gauß for determinants

For calculating $\det(A)$, you can add multiples of a row to another row or add multiples of a column to another column without changing the determinant. If you exchange two rows or two columns, you simply have to change the sign. Do not scale rows or columns since this changes the determinant.

In a formal way, we would say:

- Compute $PA = LU$, count the row permutations, to find either $\det(P) = +1$ or $\det(P) = -1$
- $\det(A) = \det(P)\det(U)$.

sign triangular matrix

Example 4.19. 5x5 (Combine Gauß and Laplace) zeros

$$A = \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \overset{R2-R5}{\underset{\text{using } Z_{2-1\cdot 5}}{\rightsquigarrow}} \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} =: B$$

det(A)

=

$$\det(B) = \det \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} = \underbrace{(-1)^{3+5}}_{=1} \cdot 1 \cdot \underbrace{\det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}}_{=:C}$$

$\det(A) = \det(C)$

$$C = \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix} \overset{C3+C4}{\underset{C2-2C4}{\rightsquigarrow}} \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix} =: D$$

det(A)

=

$$\det(D) = \det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix} = \underbrace{(-1)^{2+4}}_{=1} \cdot 2 \cdot \underbrace{\det \begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix}}_{=:E}.$$

Now, we only have a 3×3 matrix and use the formula of Sarrus:

$$\begin{aligned} \det(E) = &(-1)\cdot(-2)\cdot 3 \quad + \quad 1\cdot(-2)\cdot 1 \quad + \quad (-2)\cdot 1\cdot(-4) \\ &- \quad 1\cdot(-2)\cdot(-2) \quad - \quad (-4)\cdot(-2)\cdot(-1) \quad - \quad 3\cdot 1\cdot 1 \\ = &\, 6 - 2 + 8 - 4 + 8 - 3 = 13 \end{aligned}$$

In summary:

$$\det(A) = \det(B) = 1 \cdot \det(C) = \det(D) = 2 \cdot \det(E) = 2 \cdot 13 = \underline{\underline{26}}.$$

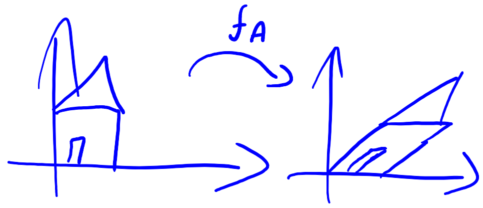
Remark:

- $\det(A^{-1}) = \frac{1}{\det(A)}$ *(if the inverse exists)*
- *If Q is an orthogonal matrix $(Q^T Q = \mathbb{1})$, then $\det(Q) = \pm 1$*
- *Let P be a row permutation matrix, then $\det(P) = 1$, if the number of row exchanges is even, and $\det(P) = -1$ if it is odd.*
- *If $PA = LU$, then $\det(A) = \frac{1}{\det(P)} \det(L)\det(U) = \det(P)\det(U) = \pm\det(U)$.*
- *If $A = S^{-1}BS$, then $\det(A) = \frac{1}{\det(S)} \det(B)\det(S) = \det(B)$ (similar matrices have the same determinant).*

Attention! Comparison: $n^3/3$ (Gauß) vs. $n!$ (Laplace/Leibniz formula)

n	2	3	4	5	6	7	8	9	10	$\cdots$	20
$n^3/3$	2	9	21	42	72	114	171	243	333	$\cdots$	2667
$n!$	2	6	24	120	720	5040	40320	362880	3628800	$\cdots$	$2.4 \cdot 10^{18}$

4.5 Determinants for linear maps



- For each matrix A, there is the linear map $f_A : \mathbb{R}^n \to \mathbb{R}^n$.
- For each linear map $f : \mathbb{R}^n \to \mathbb{R}^n$, there is a exactly one matrix A such that $f = f_A$.
- The columns of A are then the images of the unit cube under f_A.
- Then $\det(A)$ is the *relative change of volume* (of the unit cube) caused by f_A.

$\det(A)$ scaling factor of volumes

Definition 4.20. Determinant for f_A

For a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$, we define

$$\det(f) := \det(A)$$

where A it the uniquely determined matrix with $f = f_A$.

In fact $\det(f)$ is the relative change of all volumes and we remind that we have the following:

Let $A, B \in \mathbb{R}^{n \times n}$. We have the formula:

$$\det(f_A \circ f_B) = \det(f_A)\det(f_B)$$