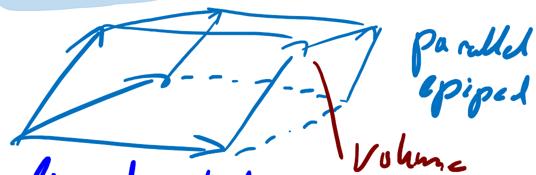
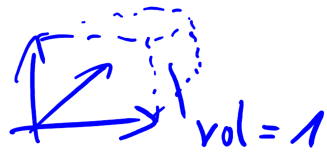
Recap: - vector space $\mathbb{R}^n$, matrices, basis, dimension
- linear maps, Gaussian elimination

4 Determinants

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. The determinant $\det(A) \in \mathbb{R}$ of A is a special real number, associated to A.



- measures "volumes"
- $\det(A) = 0 \iff$ columns are lin. dependent
  $\iff A$ is not invertible
- sign $\rightsquigarrow$ orientation, $\det(\mathbb{1}) = 1$



- Properties of a volume?

4.1 Determinant in two dimensions

We already know how to solve the system $A\mathbf{x} = \mathbf{b}$ if A is a square matrix. The determinant should then tell us if the system has a unique solution before solving it. For a 2×2 LES, we get (for $a_{11} \neq 0$):

$$\begin{array}{c} E_1 \to \\ E_2 \to \end{array} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \overset{E_2 - \frac{a_{21}}{a_{11}} E_1}{\underset{a_{11} \cdot E_2}{\rightsquigarrow}} \left(\begin{array}{cc|c} \boxed{a_{11}} & a_{12} & b_1 \\ 0 & \boxed{a_{11}a_{22} - a_{12}a_{21}} & b_2 a_{11} - b_1 a_{21} \end{array}\right)$$

$\neq 0 \iff$ unique solution
(2 pivots, $\operatorname{Ker}(A) = \{0\}$, $\operatorname{Ran}(A) = \mathbb{R}^2$)

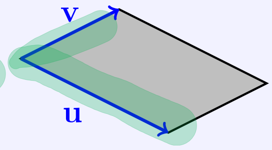
Definition 4.1. + −

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, *we call* $\det(A) := a_{11}a_{22} - a_{12}a_{21}$ *the determinant of* A.

The determinant in two dimensions has an immediate interpretation when we compare it to an area measurement:

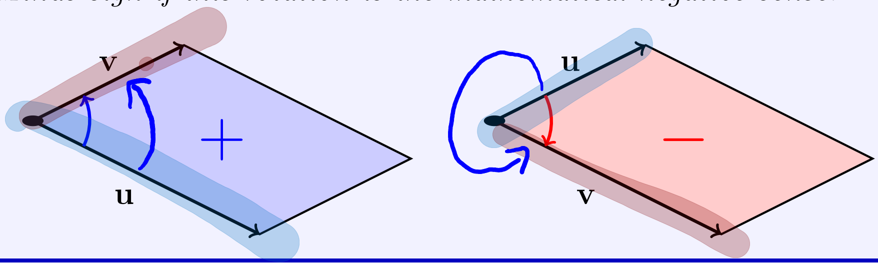
Consider two vectors $\mathbf{u} = \binom{u_1}{u_2}$ *and* $\mathbf{v} = \binom{v_1}{v_2}$ *from* $\mathbb{R}^2$ *and the spanned parallelogram. We define* $\mathrm{Area}(\mathbf{u}, \mathbf{v}) \in \mathbb{R}$ *as the real number that fulfils*

$$|\mathrm{Area}(\mathbf{u}, \mathbf{v})| = \textit{area of parallelogram}$$

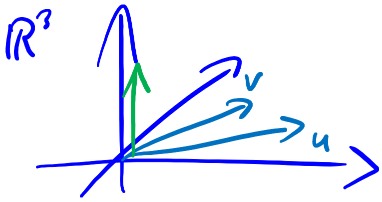


and the sign of $\mathrm{Area}(\mathbf{u}, \mathbf{v})$ *is chosen in the following way:*

- ***Plus sign*** *if then rotation by rotating* $\mathbf{u}$ *such that the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *gets smaller is the mathematical positive sense,*
- *Minus sign if this rotation is the mathematical negative sense.*



Recall Section 2.6: Area of parallelogram $\leftrightsquigarrow$ cross product (in $\mathbb{R}^3$)



x-y-plane $\leftrightsquigarrow$ $\mathbb{R}^2$

$$\tilde{\mathbf{u}} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{v}} := \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}. \qquad \Longrightarrow \qquad \tilde{\mathbf{u}} \times \tilde{\mathbf{v}} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Then we find:

$$|\mathrm{Area}(\mathbf{u}, \mathbf{v})| = \|\tilde{\mathbf{u}} \times \tilde{\mathbf{v}}\| = \left\| \begin{pmatrix} u_2 0 - 0 v_2 \\ 0 v_1 - u_1 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \right\| = \underbrace{|u_1 v_2 - u_2 v_1|}_{\det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}}.$$

Without the absolute value this coincides with the determinant of the matrix

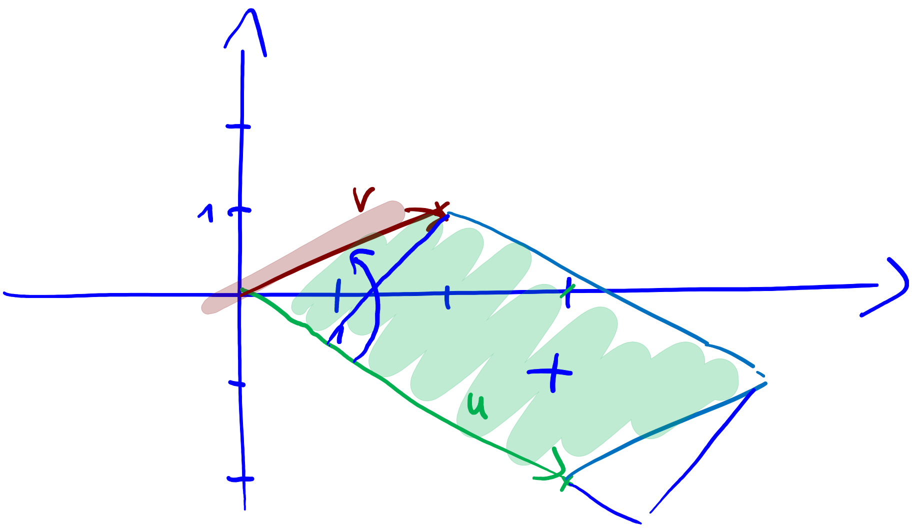
$$A := \begin{pmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{pmatrix}. \in \mathbb{R}^{2 \times 2}$$

Indeed, also the sign rule from above is fulfilled. (right-hand rule)

Proposition 4.2.

$$\mathrm{Area}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1 \tag{4.1}$$

Example 4.3. (a) If we look at $\mathbf{u} = \binom{3}{-2}$ and $\mathbf{v} = \binom{2}{1}$, we get:



$$\text{Area}(\mathbf{u}, \mathbf{v}) = \text{Area}\left(\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 3 \cdot 1 - (-2) \cdot 2 = 7.$$

The area is 7 and the orientation is positive.

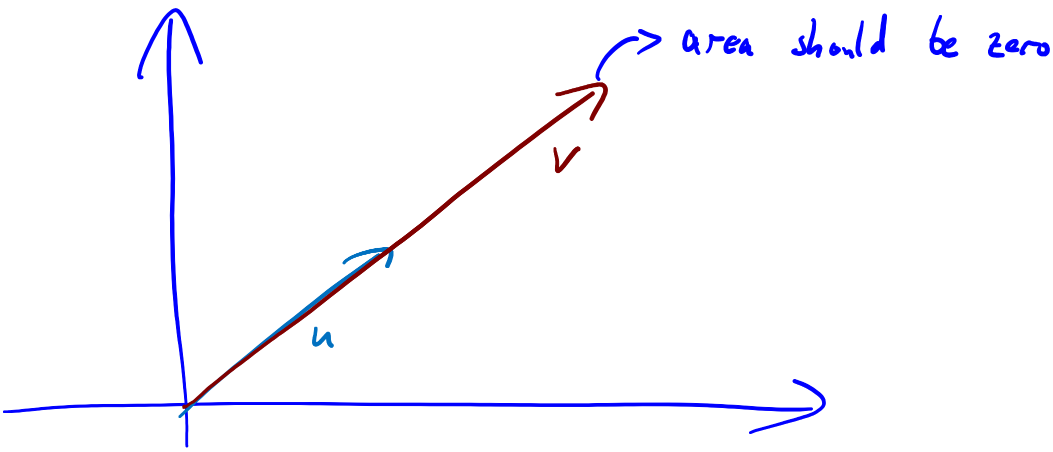
(b) The other ordering gets:

$$\text{Area}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right) = 2 \cdot (-2) - 1 \cdot 3 = -7.$$

The area is again 7 but the orientation is negative.

(c) Choose $\mathbf{u} = \binom{u_1}{u_2}$ and the scaled vector $\mathbf{v} = \alpha\mathbf{u} = \binom{\alpha u_1}{\alpha u_2}$. Then:

$$\text{Area}(\mathbf{u}, \mathbf{v}) = \text{Area}(\mathbf{u}, \alpha\mathbf{u}) = \text{Area}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}\right) = u_1 \alpha u_2 - u_2 \alpha u_1 = 0.$$



4.2 Determinant as a volume measure

in $\mathbb{R}^2$: area
in $\mathbb{R}^3$ "usual" volume
in $\mathbb{R}^1$: length

⇝ n-dim. volume
⇝ Volume function Vol_n (Vol_2 = Area)
↪ properties we expect of a volume function!

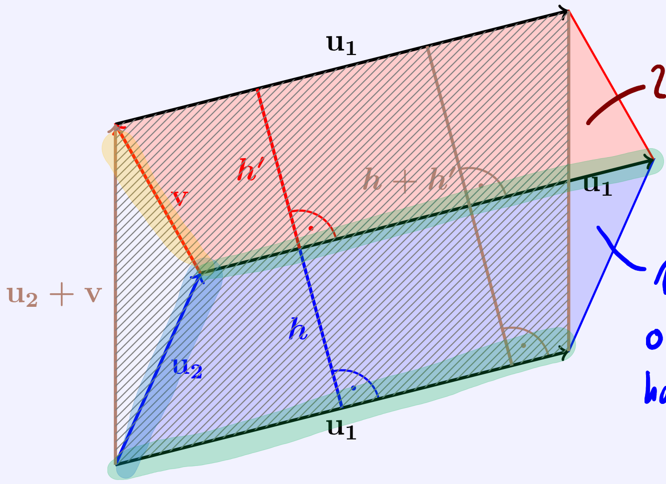
Definition 4.4. Properties that Vol_n should have.

The n-dimensional volume function $\mathrm{Vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$ *that gets n vectors as input should fulfil:*

(1) $\mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{u}_j + \mathbf{v}, \ldots, \mathbf{u}_n) = \mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{u}_j, \ldots, \mathbf{u}_n) + \mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{v}, \ldots, \mathbf{u}_n)$ *for all* $\mathbf{u}_1, \ldots \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^n$ *and* $j \in \{1, \ldots, n\}$.

in $\mathbb{R}^2$:

$\mathrm{Vol}_2(u_1, u_2 + v) = \mathrm{Vol}_2(u_1, u_2) + \mathrm{Vol}_2(u_1, v)$



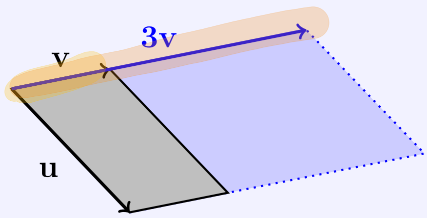
2nd part on the right-hand side

1st part on the right-hand side

(Picture in the case $n = j = 2$)

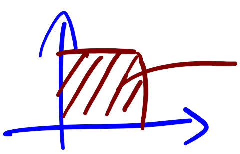
(2) $\mathrm{Vol}_n(\mathbf{u}_1, \ldots, \alpha\mathbf{u}_j, \ldots, \mathbf{u}_n) = \alpha\,\mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{u}_j, \ldots, \mathbf{u}_n)$ *for all* $\mathbf{u}_1, \ldots \mathbf{u}_n \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ *and* $j \in \{1, \ldots, n\}$.

$\mathrm{Vol}_2(u_1, 3v) = 3\,\mathrm{Vol}_2(u_1, v)$



(3) $\mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_j, \ldots, \mathbf{u}_n) = -\mathrm{Vol}_n(\mathbf{u}_1, \ldots, \mathbf{u}_j, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_n)$ *for all* $\mathbf{u}_1, \ldots \mathbf{u}_n \in \mathbb{R}^n$, *and* $i, j \in \{1, \ldots, n\}$ *with* $i \neq j$.

(4) The unit cube $(\mathbf{u}_1 = \mathbf{e}_1, \ldots, \mathbf{u}_n = \mathbf{e}_n)$ *has volume 1:* $\mathrm{Vol}(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.



vol = 1

Vol_n: (1),(2) linear in each component
(3) antisymmetric
(4) normalised

First, we show an easy consequence that follows from the two properties (2) and (3):

Proposition 4.5. Colinear vectors do not have an area.

For all $\mathbf{u} \in \mathbb{R}^2$ *and* $\alpha \in \mathbb{R}$, *we have* $\mathrm{Vol}_2(\mathbf{u}, \alpha\mathbf{u}) = 0$.

$\to 2\,\mathrm{Vol}_2(u,u) = 0$

Proof. Because of (3), we find $\mathrm{Vol}_2(\mathbf{u}, \mathbf{u}) = -\mathrm{Vol}_2(\mathbf{u}, \mathbf{u})$ and this implies $\mathrm{Vol}_2(\mathbf{u}, \mathbf{u}) = 0$. Since (2) holds, we get $\mathrm{Vol}_2(\mathbf{u}, \alpha\mathbf{u}) = \alpha\,\mathrm{Vol}_2(\mathbf{u}, \mathbf{u}) = \alpha\, 0 = 0$. □

Now, we can prove the formula (4.1):

Proposition 4.6.

If Vol_2 *fulfils (1), (2), (3), (4), then for all* $\mathbf{u} = \binom{u_1}{u_2}, \mathbf{v} = \binom{v_1}{v_2} \in \mathbb{R}^2$ *the following holds*

$$\mathrm{Vol}_2(\mathbf{u}, \mathbf{v}) = \mathrm{Vol}_2\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = +u_1 v_2 - u_2 v_1.$$

$$\begin{matrix} u_1 & v_1 \\ u_2 & v_2 \end{matrix}$$

Proof.

$$\mathrm{Vol}_2\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \mathrm{Vol}_2\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$$

$$\overset{(1)}{=} \mathrm{Vol}_2\left(\begin{pmatrix}u_1\\0\end{pmatrix}, \begin{pmatrix}v_1\\v_2\end{pmatrix}\right) + \mathrm{Vol}_2\left(\begin{pmatrix}0\\u_2\end{pmatrix}, \begin{pmatrix}v_1\\v_2\end{pmatrix}\right)$$

$$\overset{(1)}{=} \underbrace{\mathrm{Vol}_2\left(\begin{pmatrix}u_1\\0\end{pmatrix}, \begin{pmatrix}v_1\\0\end{pmatrix}\right)}_{=0} + \mathrm{Vol}_2\left(\begin{pmatrix}u_1\\0\end{pmatrix}, \begin{pmatrix}0\\v_2\end{pmatrix}\right)$$

$$+ \mathrm{Vol}_2\left(\begin{pmatrix}0\\u_2\end{pmatrix}, \begin{pmatrix}v_1\\0\end{pmatrix}\right) + \underbrace{\mathrm{Vol}_2\left(\begin{pmatrix}0\\u_2\end{pmatrix}, \begin{pmatrix}0\\v_2\end{pmatrix}\right)}_{=0 \text{ (Prop. 4.5)}}$$

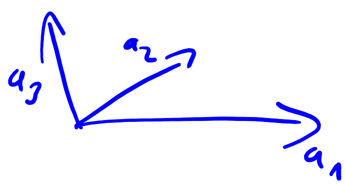
$$\overset{(2)}{=} u_1 v_2 \underbrace{\mathrm{Vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)}_{=1 \text{ by } (4)} + u_2 v_1 \underbrace{\mathrm{Vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)}_{=-1 \text{ by } (3),(4)}$$

$$= u_1 v_2 - u_2 v_1 = \mathrm{Area}\left(\begin{pmatrix}u_1\\u_2\end{pmatrix}, \begin{pmatrix}v_1\\v_2\end{pmatrix}\right) \checkmark$$

$\rightsquigarrow$ Do the same in n dimensions!

$\square$

Note that this proves that the volume function Vol_2 is uniquely defined by the four properties (1), (2), (3) and (4) alone. We expect the same for arbitrary dimension n and indeed we prove this now.



For $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \in \mathbb{R}^n$, it follows

entries

$$\mathrm{Vol}_n(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathrm{Vol}_n\big(\overbrace{a_{11}\mathbf{e}_1 + \ldots + a_{n1}\mathbf{e}_n}^{a_1}, \underbrace{\ldots, a_{1n}\mathbf{e}_1 + \ldots + a_{nn}\mathbf{e}_n}_{(*)}\big).$$

Using the linearity in each entry, we can conclude:

$$\mathrm{Vol}_n(a_1, \ldots, a_n) \overset{(1)}{=} \mathrm{Vol}_n(a_{11}e_1, (*)) + \mathrm{Vol}_n(a_{21}e_2 + \cdots + a_{n1}e_1, (*))$$

$$\overset{(2)}{=} a_{11}\mathrm{Vol}_n(e_1, (*)) + a_{21}\mathrm{Vol}_n(e_2, (*)) + \cdots$$ the same for the other ones

$$= \sum_{i_1=1}^{n} a_{i_1 1} \mathrm{Vol}_n\big(e_{i_1}, \underbrace{a_{12}e_1 + \dots + a_{n2}e_n}_{\text{Do it again!}}, \dots, a_{1n}e_1 + \dots + a_{nn}e_n\big)$$

$$\mathrm{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \underbrace{\mathrm{Vol}_n(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_n})}$$

$= 0$ if two indices coincide (by (3))

$= \pm 1$ depending on the permutation

$$= \sum_{(i_1, \dots, i_n) \in \{1,\dots,n\} \times \dots \times \{1,\dots,n\}}$$

actually, we only need to sum over $(i_1, \dots, i_n)$ where each one is different: List of the numbers $1, \dots, n$ (in any order)

$\rightsquigarrow$ permutation of the set $\{1, \dots, n\}$ $\rightsquigarrow$ $\mathcal{P}_n$ = All permutations

Example: $(1, 2, 4, 3, 5) \in \mathcal{P}_5$ is an odd permutation of $\{1, \dots, n\}$

[Alternatively: permutation: map: $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$ bijective]

For a permutation $\tau \in \mathcal{P}_n$, we define $\mathrm{sgn}(\tau) = 1$ if one can use an even number of exchanges of two elements to get from τ to $(1, 2, \dots, n)$. If one needs an odd number of exchanges of two elements to get from τ to $(1, 2, \dots, n)$, we define $\mathrm{sgn}(\tau) = -1$.

Repeatable usage of (3) shows $\mathrm{Vol}_n(\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_n}) = \mathrm{sgn}(i_1, \dots, i_n)$. In summary, we get

Proposition 4.7. Leibniz formula

The volume form Vol_n *is uniquely determined by the properties (1),(2),(3) and (4) and fulfils for* n *vectors* $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \in \mathbb{R}^n$:

$$\mathrm{Vol}_n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{(i_1,\dots,i_n) \in \mathcal{P}_n} \mathrm{sgn}(i_1, \dots, i_n) a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n}. \quad (4.2)$$

Proof. The calculation from above shows that (4.2) is the only function that fulfils all the four rules. □

In $\mathbb{R}^2$: Only two permutations in $\mathcal{P}_2$: $\overset{i_1\, i_2}{(1, 2)}$, $\overset{i_1\, i_2}{(2, 1)}$

$$\mathrm{Vol}_2(a_1, a_2) = a_{11} a_{22} - a_{21} a_{12}$$

Definition 4.8. Determinant of square matrices

For a square matrix $A \in \mathbb{R}^{n\times n}$ with entries

$$A := \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

we define the determinant as the volume measure of the column vectors:

$$\det(A) := \mathrm{Vol}_n(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \sum_{(i_1,\ldots,i_n)\in\mathcal{P}_n} \mathrm{sgn}(i_1,\ldots,i_n)\, a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n}\,.$$

Remark:

You can remember the Leibniz formula of the determinant $\det(A)$ *in the following way:*

(1) Build a product of n factors out of the entries in A. From each row and each column you are only allowed to choose one factor.

a_{35}

(2) Sum up all the possibilities for such a product where you add a minus-sign for the odd permutations.

Example 4.9. Consider the matrix $P_{k\leftrightarrow\ell}$ that we used in the Gaussian elimination to switch the kth row with the ℓth row. Let's denote the entries by p_{ij} and then we know

Example:

$$P = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$p_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i,j \neq k,\ell \\ 1 & \text{if } i = k, j = \ell \text{ or } i = \ell, j = k \\ 0 & \text{else} \end{cases}$$

$$\sum_{\tau\in\mathcal{P}_n} \mathrm{sgn}(\tau)$$

$$\det(P_{k\leftrightarrow\ell}) = \mathrm{sgn}(\tau) p_{11} \cdots p_{k,\ell} \cdots p_{\ell,k} \cdots p_{nn} = -1$$

↑ odd

Since the permutation is only one single exchange, the sign is -1. Of course, we expect this result by property (3) of the volume form.

Recall: Leibniz formula ⇝ sum up $n!$ terms

⇝ $n = 2$ $2! = 2$

$\rightsquigarrow n = 3 \qquad 3! = 6$

$\rightsquigarrow n = 4 \qquad 4! = 24$

n

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Rule of thumb: Rule of Sarrus (Only for $n = 3$)

$$\begin{array}{cccccc} & & & & & - \\ \boldsymbol{u}_1 & \boldsymbol{v}_1 & \boldsymbol{w}_1 & u_1 & v_1 & \\ \boldsymbol{u}_2 & \boldsymbol{v}_2 & \boldsymbol{w}_2 & u_2 & v_2 & \\ \boldsymbol{u}_3 & \boldsymbol{v}_3 & \boldsymbol{w}_3 & u_3 & v_3 & \\ & & & & & + \end{array}$$

$$\mathrm{Vol}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = +u_1v_2w_3 + v_1w_2u_3 + w_1u_2v_3 - u_3v_2w_1 - v_3w_2u_1 - w_3u_2v_1$$

Moreover, the sign of the three-dimensional volume can be easily seen by the right-hand-rule: