Attention!

Do not use equation E_1' anymore at this point!
Otherwise, you would bring the variable x_1 back in the game.

VL8 ↓

Gauß with a bug

We start with a square matrix $A \in \mathbb{R}^{n\times n}$. Let us write $\widetilde{A} := (A|\mathbf{b})$ as a row matrix:

$$\widetilde{A} = (A|\mathbf{b}) = \left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right) = \begin{pmatrix} - \widetilde{\boldsymbol{\alpha}}_1^T - \\ - \widetilde{\boldsymbol{\alpha}}_2^T - \\ \vdots \\ - \widetilde{\boldsymbol{\alpha}}_n^T - \end{pmatrix}$$

generate zeros

So a_{ij} is the j^{th} entry of $\widetilde{\boldsymbol{\alpha}}_i^T$.

We can eliminate a_{21} by adding rows: $\widetilde{\boldsymbol{\alpha}}_2^T \rightsquigarrow \widetilde{\boldsymbol{\alpha}}_2^T - \lambda_2 \widetilde{\boldsymbol{\alpha}}_1^T$, where $\lambda_2 = \frac{a_{21}}{a_{11}}$.

This can be written in terms of a matrix, and we obtain:

$$Z_{2-\lambda_2 1}(A|\mathbf{b}) = \left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right) = \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1^T \\ \widetilde{\boldsymbol{\alpha}}_2^T - \lambda_2 \widetilde{\boldsymbol{\alpha}}_1^T \\ \widetilde{\boldsymbol{\alpha}}_3^T \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_n^T \end{pmatrix}$$

do this for all rows

Now we can do the same with all other rows, defining $\lambda_i = \frac{a_{i1}}{a_{11}}$, and computing:

$$\underbrace{Z_{n-\lambda_n 1} \dots Z_{2-\lambda_2 1}}_{L_1^{-1}}(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & \boxed{\tilde{a}_{22}} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \tilde{a}_{n2} & \dots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right) = \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1^T \\ \widetilde{\boldsymbol{\alpha}}_2^T - \lambda_2 \widetilde{\boldsymbol{\alpha}}_1^T \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_n^T - \lambda_n \widetilde{\boldsymbol{\alpha}}_1^T \end{pmatrix} = L_1^{-1}(A|\mathbf{b})$$

Since L_1^{-1} substracts the first row from all others, its inverse is easily seen to be the matrix that adds this row to all others:

$$L_1^{-1} = \begin{pmatrix} 1 & & & \\ -\lambda_2 & 1 & & \\ \vdots & & \ddots & \\ -\lambda_n & & & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & & & \\ \lambda_2 & 1 & & \\ \vdots & & \ddots & \\ \lambda_n & & & 1 \end{pmatrix}$$

Once, we have eliminated the entries $a_{21} \dots a_{n1}$. We can do the same with $\tilde{a}_{32} \dots \tilde{a}_{n2}$. Here we use factors $\tilde{\lambda}_i = \frac{\tilde{a}_{i2}}{\tilde{a}_{22}}$

We then obtain:

$$L_2^{-1}L_1^{-1}(A|b) = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & \boxed{\tilde{a}_{22}} & \tilde{a}_{23} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ 0 & 0 & \hat{a}_{33} & \dots & \hat{a}_{3n} & \hat{b}_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \hat{a}_{n3} & \dots & \hat{a}_{nn} & \hat{b}_n \end{array}\right)$$

It (luckily) turns out, that

$$L_1L_2 = \begin{pmatrix} 1 & & & & \\ \lambda_2 & 1 & & & \\ \lambda_3 & \tilde{\lambda}_3 & 1 & & \\ \vdots & \vdots & & \ddots & \\ \lambda_n & \tilde{\lambda}_n & & & 1 \end{pmatrix}$$

If we do this column by column, (and don't run out of hats) we obtain:

$$\underbrace{L_{n-1}^{-1}\dots L_1^{-1}}_{L^{-1}}(A|\mathbf{b}) = \left(\begin{array}{ccc|c} u_{11} & \dots & u_{1n} & c_1 \\ & \ddots & \vdots & \vdots \\ & & u_{nn} & c_n \end{array}\right) = (U|\mathbf{c}).$$

zeros

where L is unit lower triangular:

$L^{-1}A = U$

$L^{-1}b = c$

$$L = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ \vdots & & \ddots & \ddots & \\ l_{n1} & \dots & & l_{n(n-1)} & 1 \end{pmatrix}$$

LU-decomposition $A = LU$

We thus have $L^{-1}A = U$, and $L^{-1}\mathbf{b} = \mathbf{c}$. Multiplication by L yields the famous *LU-decomposition*:

$$A = LU, \qquad \mathbf{b} = L\mathbf{c}.$$

Here L is lower triangular, and U is upper triangular. Once, we have decomposed A, we can, for given $\mathbf{b}$ compute $\mathbf{x}$ as follows:

solve $L\mathbf{c} = \mathbf{b}$ "forward substitution"
solve $U\mathbf{x} = \mathbf{c}$ "backward substitution"

$$\downarrow \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Solve: $Ux = L^{-1}b$

Then $A\mathbf{x} = LU\mathbf{x} = L\mathbf{c} = \mathbf{b}$.

$$\underbrace{\begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ & & \ddots & \\ l_{n1} & \cdots & & 1 \end{pmatrix}}_{\text{remembers the steps of the row operations}} \underbrace{\begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & \ddots & \vdots \\ & & u_{nn} \end{pmatrix}}_{\text{upper triangular}} = A$$

Gaussian elimination without pivoting ($A \in \mathbb{R}^{n\times n}$)

$(U|\mathbf{c}) = (A|\mathbf{b})$, $L = \mathbb{1}_n$
for $j = 1 \dots n$ (loop over columns)
  for $i = j+1 \dots n$ (loop over rows)
    $l_{ij} = \frac{u_{ij}}{u_{jj}}$
    $u_{ij} = 0$ (eliminate entry)
    for $s = j+1 \dots n$
      $u_{is} = u_{is} - l_{ij}u_{js}$ (subtract remaining entries)
    $c_i = c_i - l_{ij}c_j$ (subtract rhs)

- We recognise three nested loops, and thus, the cost of this algorithm is proportional to n^3.
- After that, we have to perform backward substitution to compute $\mathbf{x}$ from $\mathbf{c}$.
- If $\mathbf{b}$ is only known after the decomposition, we can compute $\mathbf{c}$ by forward substitution.
- In computer libraries A is overwritten by L and U. The upper triangular part is used to store U, the lower triangular part is used to store L:

$$\begin{matrix} u_{11} & \cdots & \cdots & u_{1n} \\ l_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ l_{n1} & \cdots & l_{n(n-1)} & u_{nn} \end{matrix} = L\backslash U$$

This is called *in place factorisation*. It is possible, since we know anyway that U is zero below the diagonal, L is zero above the diagonal, and $l_{ii} = 1$. We may call this *storage matrix* $L\backslash U$.

Example 3.51. ***LU*-factorisation:**

We do the row operation and save them in the matrix L:

$$A = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}\begin{pmatrix} 2 & 1 & 1 \\ 4 & 4 & 6 \\ -2 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & \boxed{2} & 4 \\ 0 & 4 & 10 \end{pmatrix}\begin{array}{c} E_1' \\ E_2' \\ E_3' \end{array} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}}_{=:L} \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}}_{=:U}$$

$E_2 - 2E_1$, $E_3 - (-1)E_1$ $E_3' - 2E_2'$

Or if one uses the storage-saving notation:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 4 & 6 \\ -2 & 3 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \rightsquigarrow L\backslash U = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & 2 & 2 \end{pmatrix},$$

For a right-hand side $\mathbf{b}$*, one simply does the same row calculation steps:*

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} =: \mathbf{c}$$

Alternatively: $\mathbf{c} := L^{-1}\mathbf{b}$*. For finding then the (unique) solution of* $A\mathbf{x} = \mathbf{b}$*, just compute* $\mathbf{x} = U^{-1}\mathbf{c}$ *as usual.*

The bug:

- What, if u_{jj} is zero at some stage of the computation? Then we have division by 0.
- Also, on the computer, if u_{jj} is very small, say 10^{-14}, then due to round-off error, problems may also occur.

Remark: Gaussian elimination or *LU*-decomposition?

For solving a system $A\mathbf{x} = \mathbf{b}$ *you have now two options:*

(a) Gaussian elimination of $(A|\mathbf{b})$ *without memorising the row operations.*

(b) LU-decomposition of A with memorising the row operations in the matrix L.

If you are just interested in the solution(s) of a given LES, then you will just do the Gaussian elimination step by step until you reach the upper triangle form (or the row echelon form, see next section).

Example 3.52. Let us a look at a higher dimensional and non-square example:

$$\begin{array}{l|rcrcrcrcrcr|} \cline{2-12} E_1\colon & x_1 & + & 2x_2 & & & + & x_4 & & & = & 3 \\ E_2\colon & 4x_1 & + & 8x_2 & + & 2x_3 & + & 3x_4 & + & 4x_5 & = & 14 \\ E_3\colon & & & & & 2x_3 & + & 3x_4 & + & 12x_5 & = & 10 \\ E_4\colon & -3x_1 & - & 6x_2 & - & 6x_3 & + & 8x_4 & + & 4x_5 & = & 4 \\ \cline{2-12} \end{array} \tag{3.7}$$

pivot

You should immediately rewrite this in an augmented matrix form:

$$
(A|\mathbf{b}) = \begin{array}{r} E_1: \\ E_2: \\ E_3: \\ E_4: \end{array}
\left(\begin{array}{rrrrr|r}
\boxed{1} & 2 & 0 & 1 & 0 & 3 \\
4 & 8 & 2 & 3 & 4 & 14 \\
0 & 0 & 2 & 3 & 12 & 10 \\
-3 & -6 & -6 & 8 & 4 & 4
\end{array}\right)
$$

The entry in grey $\boxed{1}$ is first one we have to consider. All entries below should get zero after the first elimination.

- multiply $\frac{4}{\boxed{1}} = 4$ to E_1 and subtract the result from E_2,
- multiply $\frac{-3}{\boxed{1}} = (-3)$ to E_1 and subtract the result from E_4.

$$
\begin{array}{r|c|l}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} & \\
\begin{array}{r} E_1: \\ E_2: \\ E_3: \\ E_4: \end{array} &
\left(\begin{array}{rrrrr|r}
\boxed{1} & 2 & 0 & 1 & 0 & 3 \\
4 & 8 & 2 & 3 & 4 & 14 \\
0 & 0 & 2 & 3 & 12 & 10 \\
-3 & -6 & -6 & 8 & 4 & 4
\end{array}\right) &
\begin{array}{l} \\ -4 \cdot E_1 \\ \\ +3 \cdot E_1 \end{array}
\end{array}
\rightsquigarrow
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 2 & 3 & 12 & 10 \\
0 & 0 & -6 & 11 & 4 & 13
\end{array}\right)
$$

The next number on the diagonal is a zero and it seems like that our algorithm has to stop here. However, since below there are also zeros, the column is already eliminated. We can just ignore the variable x_2 at this point and just restart the algorithm with starting point $\boxed{2}$.

Just subtract the equation E_2' with the right factor from the other rows: $\frac{2}{\boxed{2}} = 1$ times from E_3' and $\frac{-6}{\boxed{2}} = (-3)$ times from E_4'. We get:

$$
\begin{array}{rcl}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} & \\
\begin{array}{r} E_1': \\ E_2': \\ E_3': \\ E_4': \end{array} &
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & \boxed{2} & -1 & 4 & 2 \\
0 & 0 & 2 & 3 & 12 & 10 \\
0 & 0 & -6 & 11 & 4 & 13
\end{array}\right) &
\begin{array}{l} \\ \\ -1 \cdot E_2' \\ +3 \cdot E_2' \end{array}
\end{array}
\rightsquigarrow
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 8 & 16 & 19
\end{array}\right)
$$

Next variable is x_4. Now we consider $\boxed{4}$. Multiply E_3'' with $\frac{8}{\boxed{4}} = 2$ and subtract from equation E_4'':

$$
\begin{array}{rcl}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} & \\
\begin{array}{r} E_1'': \\ E_2'': \\ E_3'': \\ E_4'': \end{array} &
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & \boxed{4} & 8 & 8 \\
0 & 0 & 0 & 8 & 16 & 19
\end{array}\right) &
\begin{array}{l} \\ \\ \\ -2 \cdot E_3'' \end{array}
\end{array}
\rightsquigarrow
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right)
$$

Now, we cannot use any rows for elimination and we are finished. We get the following result:

$$
\begin{array}{r}
\\
E_1'''': \\
E_2'''': \\
E_3'''': \\
E_4'''':
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right)
\end{array}
\tag{3.8}
$$

$\rightsquigarrow$ no solution!

$\rightsquigarrow 0 \cdot x_5 = 3$

This is not a triangle matrix like in Example 3.42 but an upper triangle matrix by definition since below the diagonal, there are just zeros. This form is called the row echelon form and defined below.

3.11.4 Row echelon form

Definition 3.53. Row echelon form, pivot element

A matrix $A \in \mathbb{R}^{m \times n}$ in the form of the left-hand side of (3.8) is called row echelon form. This means that the matrix A fulfils:

- *all zero rows, if any, are at the bottom of the matrix,*
- *for each row: the first nonzero number from the left is always strictly to the right of the first nonzero coefficient from the row above it.*

This leading nonzero number in each row is called the pivot.

In the row echelon form we can put the variable into two groups:

Definition 3.54. Free and leading variables

Variables in the column of a pivot are called leading variables.
The other variables are called free variables.

Example 3.55. Looking at equation (3.8) again, we can distinguish the variables

$$
\begin{array}{r}
\\
G_1'''': \\
G_2'''': \\
G_3'''': \\
G_4'''':
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right)
\end{array}
\tag{3.9}
$$

In this example x_1, x_3 and x_4 are the leading variables and x_2 and x_5 are free.

- *Free variables can be chosen independently in $\mathbb{R}$.*
- *The leading variables are chosen dependently of the free variables.*

LES $Ax = b \rightsquigarrow (A|b) \xrightarrow{\text{row operations}} (A', b')$ row echelon form

$\rightsquigarrow$ putting the free variables to the right $\rightsquigarrow$ backward substitution

$\left.\begin{array}{l} \dim(\text{Ran}(A)) = 3 \\ \dim(\text{Ker}(A)) = 2 \end{array}\right\} 5$

Example 3.56.

The LES

$$\begin{array}{c} \\ E_1: \\ E_2: \\ E_3: \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \end{array}\right) \end{array} \tag{3.10}$$

$\rightarrow 2x_3 - 1x_4 + \cancel{4x_5} = 2$

is already in row echelon form and can be equivalently written as

$$\begin{array}{c} \\ E_1: \\ E_2: \\ E_3: \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\ \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 - 2\,x_2 \\ 0 & \boxed{2} & -1 & 2 - 4\,x_5 \\ 0 & 0 & \boxed{4} & 8 - 8\,x_5 \end{array}\right) \end{array} \tag{3.11}$$

$\rightarrow x_1 = \dots$

$\rightsquigarrow 2x_3 - 1x_4 = 2 - 4x_5 \rightsquigarrow x_3 = \dots$

$\rightarrow 4x_4 = 8 - 8x_5$

and backward substitution gives us the solution set:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} : x_2, x_5 \in \mathbb{R} \right\}$$

Corollary 3.57.

If $A \in \mathbb{R}^{m \times n}$ is a row echelon matrix, then $\text{rank}(A)$ *is the number of leading variables and* $\dim(\text{Ker}(A))$ *is the number of free variables.*

Proof. Obviously, the columns with pivots are linearly independent vectors where the columns with free variables are a linear combination of the other ones. □

In the next section, we will generalise what we did in the example before.

3.11.5 Gaussian elimination with pivoting and $PA = LK$ decomposition

permutation matrix

row echelon form

Now we consider the general case of $A\mathbf{x} = \mathbf{b}$ with non-square $A \in \mathbb{R}^{m \times n}$. Here the role of the upper triangular U is played by a matrix K in *row echelon form*, see Definition 3.53.

Need : $P_{i \leftrightarrow j}$

Pen-and-paper strategy: "non-zero pivoting"

$$\begin{pmatrix} 0 & 0 & 2 & 3 \\ 4 & 1 & 6 & 7 \\ 0 & 0 & 2 & 3 \end{pmatrix} \underset{P_{1\leftrightarrow 2}}{\overset{\text{switch the rows}}{\rightsquigarrow}} \begin{pmatrix} 4 & 1 & 6 & 7 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Search for non-zero entry

pivot

go to the next column $\rightsquigarrow$

Remark: Pivot search

In other words: If the next entry that we want to choose as a pivot is zero, we just search the rest of the column below for a non-zero entry and switch the rows.

Example 3.58. Invertible matrix (with pivoting):

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \underbrace{(P_{2\leftrightarrow 3})^2}_{=\mathbb{1}} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \underbrace{(P_{2\leftrightarrow 3})^2}_{=\mathbb{1}} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= P_{2\leftrightarrow 3} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

exchanged!

$$P_{1\leftrightarrow 2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} P_{2\leftrightarrow 3}$$

Or if one uses the storage-saving notation:

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \overset{(P_{2\leftrightarrow 3})}{\rightsquigarrow} \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Example 3.59. Invertible matrix (with hindsight and pivoting)

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 9 \\ 2 & 4 & 6 \end{pmatrix} \overset{(P_{2\leftrightarrow 3})}{\rightsquigarrow} \begin{pmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \\ 4 & 6 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

- Doesn't matter when the row exchange happens
- Just remember which ones in which order: P

$$Ax = b \rightsquigarrow PAx = Pb \rightsquigarrow LUx = Pb$$

Gaussian elimination with pivoting ($A \in \mathbb{R}^{m\times n}$)

$K = A$, $L = \mathbb{1}_m$, $\mathbf{c} = \mathbf{b}$, $r = 1$, $P_{row} = \mathbb{1}_m$
for $j = 1 \dots n$ (loop over columns)
  perform pivot search for the first non-zero element of K at or below k_{rj}
  if i_{pivot} was found, exchange row r and row i_{pivot} of $L\backslash K$, $\mathbf{c}$, and P_{row}
    for $i = r \dots m$ (loop over rows)
      $l_{ir} = \frac{k_{ij}}{k_{rj}}$
      $k_{ij} = 0$
      for $s = r + 1 \dots m$
        $k_{is} = k_{is} - l_{ir}k_{rs}$
      $c_r = c_r - l_{ir}c_r$
    $r = r + 1$ consider the next row

Example 3.60. non-square matrix (no pivoting needed here)

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}}_{L} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1&0&0\\1&1&0\\2&1&1\end{pmatrix}}_{L} \underbrace{\begin{pmatrix}1&2&1&2\\0&0&1&1\\0&0&0&0\end{pmatrix}}_{\text{row echelon}}$$

We observe that $\operatorname{rank}(A) = 2$.

Example 3.61. Modified Example: with pivoting

$$\begin{pmatrix}1&2&1&2\\1&2&1&3\\2&4&3&5\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\2&0&1\end{pmatrix}\begin{pmatrix}1&2&1&2\\0&0&0&1\\0&0&1&1\end{pmatrix}$$
$$= P_{2\leftrightarrow 3}\begin{pmatrix}1&0&0\\2&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&2&1&2\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

The result of pen-and-paper pivoting

Although this exchange of rows happens during the course of elimination, the outcome of the resulting algorithm for matrices A can be written as

$$P_{row}A = LK,$$

where in P_{row} all the performed permutations, and K is in row echelon form. Hence, for a right hand side $\mathbf{b}$ we may solve $A\mathbf{x} = \mathbf{b}$ as follows:

$$\begin{aligned} \mathbf{w} &= P_{row}\mathbf{b} && \text{(row permutations)}\\ L\mathbf{c} &= \mathbf{w} && \text{(forward substitution)}\\ K\mathbf{x} &= \mathbf{c} && \text{(backward substitution)}\end{aligned}$$

Then $P_{row}A\mathbf{x} = LK\mathbf{x} = L\mathbf{c} = \mathbf{w} = P_{row}\mathbf{b}$. Usually, P_{row} is not stored as a matrix, but rather as a vector $\mathbf{p}$ of indices: $w_i = b_{p_i}$.

Example 3.62. We look at the example:

$$\begin{array}{l|rcrcrcrcrcr|}\hline E_1: & & & & & 2x_3 & + & 3x_4 & + & 12x_5 & = & 10\\ E_2: & 4x_1 & + & 8x_2 & + & 2x_3 & + & 3x_4 & + & 4x_5 & = & 14\\ E_3: & x_1 & + & 2x_2 & & & + & x_4 & & & = & 3\\ E_4: & -3x_1 & - & 6x_2 & - & 6x_3 & + & 8x_4 & + & 4x_5 & = & 1\\ \hline\end{array} \tag{3.12}$$

As always:

$$(A|\mathbf{b}) \quad = \quad \begin{array}{l}E_1:\\E_2:\\E_3:\\E_4:\end{array}\left(\begin{array}{rrrrr|r}0&0&2&3&12&10\\4&8&2&3&4&14\\1&2&0&1&0&3\\-3&-6&-6&8&4&1\end{array}\right)$$

We need a row exchange.

Let us exchange E_1 with E_3:

$$\begin{array}{l}E_1:\\E_2:\\E_3:\\E_4:\end{array}\left(\begin{array}{rrrrr|r}0&0&2&3&12&10\\4&8&2&3&4&14\\1&2&0&1&0&3\\-3&-6&-6&8&4&1\end{array}\right) \quad\rightsquigarrow\quad \begin{array}{l}E_1':\\E_2':\\E_3':\\E_4':\end{array}\left(\begin{array}{rrrrr|r}1&2&0&1&0&3\\4&8&2&3&4&14\\0&0&2&3&12&10\\-3&-6&-6&8&4&1\end{array}\right)$$

Now, there is a gray 1 that we will use for the subtraction:

- Subtract $\frac{4}{1} = 4$ times E_1' from E_2',
- subtract $\frac{-3}{1} = (-3)$ times E_1' from E_4'.

Here the solution:

$$\begin{array}{r} \\ E_1'\colon \\ E_2'\colon \\ E_3'\colon \\ E_4'\colon \end{array}
\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 4 & 8 & 2 & 3 & 4 & 14 \\ 0 & 0 & 2 & 3 & 12 & 10 \\ -3 & -6 & -6 & 8 & 4 & 1 \end{array}\right) \end{array}
\begin{array}{l} \\ \\ -4\cdot E_1' \\ \\ +3\cdot E_1' \end{array} \quad \rightsquigarrow$$

Also, x_2 remains only in row 1. Hence, we do not have to do anything with x_2. We can go to x_3.

There the gray 2 in E_2'' is the next pivot. Subtract E_2'' with the right multiple ($\frac{2}{2} = 1$) from E_3''. Also subtract $\frac{-6}{2} = (-3)$ times E_2'' from E_4''. We get:

$$\begin{array}{r} \\ E_1''\colon \\ E_2''\colon \\ E_3''\colon \\ E_4''\colon \end{array}
\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 2 & 3 & 12 & 10 \\ 0 & 0 & -6 & 11 & 4 & 10 \end{array}\right) \end{array}
\begin{array}{l} \\ \\ \\ -1\cdot E_2'' \\ +3\cdot E_2'' \end{array} \quad \rightsquigarrow$$

Look at x_4. Here, 4 is the pivot. We subtract E_3''' $\frac{8}{4} = 2$ times from E_4''':

$$\begin{array}{r} \\ G_1'''\colon \\ G_2'''\colon \\ G_3'''\colon \\ G_4'''\colon \end{array}
\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 0 & 4 & 8 & 8 \\ 0 & 0 & 0 & 8 & 16 & 16 \end{array}\right) \end{array}
\begin{array}{l} \\ \\ \\ \\ -2\cdot G_3''' \end{array} \quad \rightsquigarrow$$

The elimination algorithm ends. This is the wanted solution in row echelon form

$$\begin{array}{r} \\ E_1''''\colon \\ E_2''''\colon \\ E_3''''\colon \\ E_4''''\colon \end{array}
\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 0 & 4 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \tag{3.13}$$

Can you write down the set of all solutions $\mathcal{S}$?

3.12 Looking at columns and maps

$$A\mathbf{x} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \ldots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix}$$

$$\operatorname{Ran}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \{x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n \;:\; x_1, \ldots, x_n \in \mathbb{R}\} \subset \mathbb{R}^m. \quad (3.14)$$

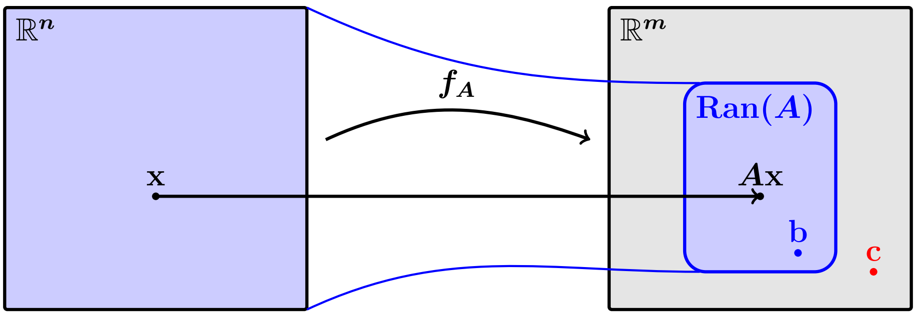
Corollary 3.63. Solvability in the column picture

For a matrix $A \in \mathbb{R}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$ the following claims are equivalent

(i) $A\mathbf{x} = \mathbf{b}$ *has at least one solution,*

(ii) $\mathbf{b} \in \operatorname{Ran}(A)$*,*

(iii) $\mathbf{b}$ *can be written as a linear combination of the columns from* A*.*



$$\mathbf{b} \in \mathbf{Ran}(A) \quad \Rightarrow \quad A\mathbf{x} = \mathbf{b} \text{ has at least one solution}$$
$$\mathbf{c} \notin \mathbf{Ran}(A) \quad \Rightarrow \quad A\mathbf{x} = \mathbf{c} \text{ has } \underline{\text{no}} \text{ solution}$$

Example 3.64. Let $A = \left(\begin{smallmatrix} 3 & 6 \\ 1 & 2 \end{smallmatrix}\right)$.Then $A\mathbf{x} = \mathbf{b}$ has at least one solution if and only if

$\mathbf{b} \in \operatorname{Ran}(A)$

This means that $\mathbf{b} \in \mathbb{R}^2$ lies on the line through $\binom{0}{0}$ and $\binom{3}{1}$.

Remember that for each matrix A there is a linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$, cf. section 3.3, defined by

$$\mathbf{x} \in \mathbb{R}^n \overset{f_A}{\mapsto} A\mathbf{x} \in \mathbb{R}^m.$$

$$\boxed{f_A(\mathbb{R}^n) = \{f_A(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\} = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \operatorname{Ran}(A).}$$

Hence, we find the following:

Proposition 3.65. Unconditional solvability needs surjectivity of f_A)

For a matrix $A \in \mathbb{R}^{m\times n}$ the following claims are equivalent:

(i) *The LES $A\mathbf{x} = \mathbf{b}$ has for* ***every*** $\mathbf{b} \in \mathbb{R}^m$ ***at least one*** *solution* $\mathbf{x}$.

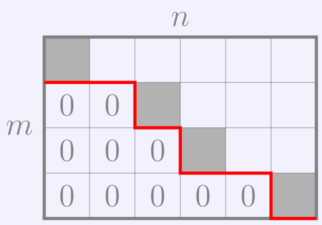
(ii) ***All*** $\mathbf{b} \in \mathbb{R}^m$ *lie in* $\mathrm{Ran}(A)$.

(iii) $\mathrm{Ran}(A) = \mathbb{R}^m$.

(iv) $\mathrm{rank}(A) = m \leq n$.

(v) *The row echelon form of A, denoted by A', has a pivot in* ***every row****.*

(vi) *f_A is surjective.*



Row echelon form A' of A:

- *each row has a pivot*
- *There are no zero rows in A'.*
- *We will never have $(0 \ \cdots \ 0 \mid c \neq 0)$ in the last row.*

Example 3.66. Consider a 3×5 matrix A and calculate the row echelon form A':

$$A = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ -1 & 2 & -2 & -2 & 3 \\ -3 & 0 & -4 & -3 & 8 \end{pmatrix} \quad \rightsquigarrow \quad \cdots \quad \rightsquigarrow \quad A' = \begin{pmatrix} \boxed{1} & 4 & 0 & 2 & -1 \\ 0 & \boxed{6} & -2 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 1 \end{pmatrix}$$

Each row of A' has a pivot and (v) from Proposition 3.65 holds. One immediately sees $\mathrm{rank}(A) = 3 = m \leq n = 5$.

(i) says that the LES $A\mathbf{x} = \mathbf{b}$ has for every right-hand sides $\mathbf{b} \in \mathbb{R}^3$ at least one solution.

Now we go to the uniqueness

Proposition 3.67. Unique solution (injectivity of f_A)

For a matrix $A \in \mathbb{R}^{m\times n}$ the following claims are equivalent:

(i) $A\mathbf{x} = \mathbf{b}$ *has for **every** $\mathbf{b} \in \mathbb{R}^m$ **at most** one solution* $\mathbf{x}$.

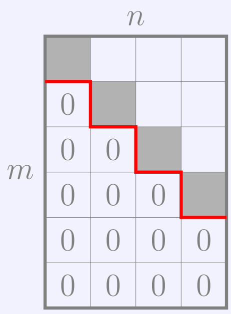
(ii) $A\mathbf{x} = \mathbf{o}$ *has only the solution* $\mathbf{x} = \mathbf{o}$.

(iii) $\mathrm{Ker}(A) = \{\mathbf{o}\}$.

(iv) $\mathrm{rank}(A) = n \leq m$.

*(v) The row echelon form of A, denoted by A', has in **every column** a pivot.*

(vi) f_A is injective.



Row echelon form A' of A:

- *Each column has a pivot.*
- *All variables are leading variables.*
- *There are no free variables.*
- *It is not possible to have more than one solution.*

Example 3.68. Consider a 4×3 matrix A and calculate the row echelon form A':

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 2 & 2 & 5 \\ -4 & -5 & -3 \\ 4 & 7 & 1 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow A' = \begin{pmatrix} \boxed{2} & 3 & 0 \\ 0 & \boxed{-1} & 5 \\ 0 & 0 & \boxed{-8} \\ 0 & 0 & 0 \end{pmatrix}$$

Each column in A' has a pivot. One also sees: $\mathrm{rank}(A) = 3 = n \leq m = 4$.

The LES $A\mathbf{x} = \mathbf{b}$ has exactly the solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ -4 \\ 4 \end{pmatrix}$; but for $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ there is no solution $\mathbf{x}$.

Both things together:

Proposition 3.69. Existence and Uniqueness of a solution

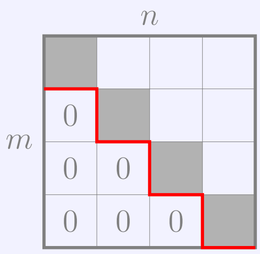
For a matrix $A \in \mathbb{R}^{m\times n}$ the following claims are equivalent:

(i) *The LES $A\mathbf{x} = \mathbf{b}$ has **for every** $\mathbf{b} \in \mathbb{R}^m$ a **unique** solution.*

(ii) $\mathrm{Ker}(A) = \{\mathbf{o}\}$ *and* $\mathrm{Ran}(A) = \mathbb{R}^m$.

(iii) $\mathrm{rank}(A) = m = n$, *i.e. A is quadratic with maximal rank.*

(iv) *f_A is bijective.*



Row echelon form A' of A:

- *Each column and row has a pivot.*
- *The matrix has to be quadratic.*
- *We have $\mathrm{rank}(A) = m = n$.*
- *The row echelon form A' has triangle form.*

Proposition 3.70. $m{=}n$**: square matrices**

For a quadratic $A \in \mathbb{R}^{n\times n}$ the following claims are equivalent:

(i) *The LES $A\mathbf{x} = \mathbf{b}$ has a solution for **for every** $\mathbf{b} \in \mathbb{R}^n$.*

(ii) *The LES $A\mathbf{x} = \mathbf{b}$ has for some $\mathbf{b} \in \mathbb{R}^n$ a **unique** solution.*

(iii) *The LES $A\mathbf{x} = \mathbf{b}$ has a unique solution **for every** $\mathbf{b} \in \mathbb{R}^n$.*

(iv) $\mathrm{Ker}(A) = \{\mathbf{o}\}$

(v) $\mathrm{Ran}(A) = \mathbb{R}^n$.

(vi) $\mathrm{rank}(A) = n$.

(vii) *For A, the row echelon form A' has a pivot in each **row**.*

(viii) *For A, the row echelon form A' has a pivot in each **column**.*

(ix) *f_A is surjective.*

(x) *f_A is injective.*

(xi) *f_A is bijective.*

Box 3.71. Fredholm alternative

For square matrices, we have either both claims below or neither of them:

- *unconditional solvability (f_A ist surjective),*
- *unique solutions ($\mathrm{Ker}(A) = \{\mathbf{o}\}$, hence f_A is injective)*

Summary

- By $\mathbb{R}^{m\times n}$ we denote number tables with m rows and n columns.
- We call these number tables *matrices* and can naturally scale them and add them Both operations in $\mathbb{R}^{m\times n}$ are realised by doing these inside the components.
- *Linear equations* look like

$$\text{constant} \cdot x_1 \; + \; \text{constant} \cdot x_2 \; + \; \cdots \; + \; \text{constant} \cdot x_n \; = \; \text{constant}\,.$$

- Systems of linear equations (LES) are finitely many of these linear equations.
- A *solution* of the system is a choice of all unknowns $x_1, \ldots, x_n$ such that all equations are satisfied.
- A short notation for LES is the matrix notation: $A\mathbf{x} = \mathbf{b}$.
- This notation leads us to the general *matrix product*.
- Each matrix A induces a *linear map* $f_A : \mathbb{R}^n \to \mathbb{R}^m$. A linear map satisfies two properties $(\,\cdot\,)$ and $(+)$.
- If f_A is bijective, the corresponding matrix is *invertible* with respect to the matrix product.
- *Linearly independent vectors* are the most efficient method to describe a linear subspace.
- A linearly independent family that generates the whole subspace U is called a *basis* of U.
- Range, rank and kernel are important objects for matrices.
- For solving a LES, we use *Gaussian elimination* or equivalently LU-decomposition. In the general case the upper diagonal matrix U is substituted by a *row echelon form*.
- Solvability and unique solvability can be equivalently formulated and, for example, read from the row echelon form.