$\to$ + and scalar multiplication

linear subspaces in $\mathbb{R}^n$

Example 2.14. The vector space $\mathbb{R}^n$ is spanned by the n unit vectors:

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \ \ldots, \ \mathbf{e}_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \ \mathbf{e}_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

because $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$ for all $\mathbf{v} \in \mathbb{R}^n$. In short: $\mathbb{R}^n = \operatorname{Span}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

VL3 ↓

all possible linear combinations

Proposition 2.15. Span is smallest linear subspace

Let $U \subset \mathbb{R}^n$ be a linear subspace and $M \subset U$ any set. Then $\operatorname{Span}(M)$ is a linear subspace and $\operatorname{Span}(M) \subset U$.

Proof: Exercise! Showing $\operatorname{Span}(M)$ is a linear subspace:

$$\left.\begin{array}{lcl} x, y \in \operatorname{Span}(M) & \Rightarrow & x + y \in \operatorname{Span}(M) \\ x \in \operatorname{Span}(M),\ \lambda \in \mathbb{R} & \Rightarrow & \lambda \cdot x \in \operatorname{Span}(M) \end{array}\right\} \operatorname{Span}(M) \text{ is a linear subspace.}$$

Fill in the details

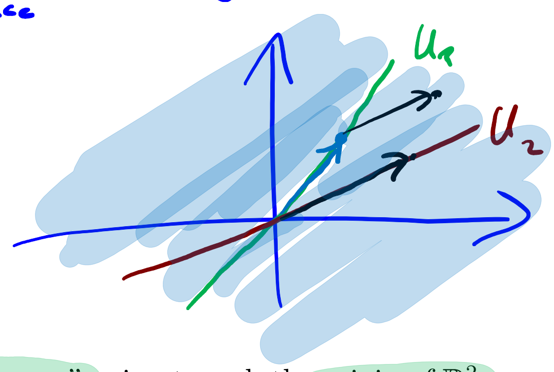
Definition 2.16. Addition for subspaces?

If U_1 and U_2 are linear subspaces in $\mathbb{R}^n$, then one defines

$$U_1 + U_2 := \operatorname{Span}(U_1 \cup U_2).$$

linear subspace

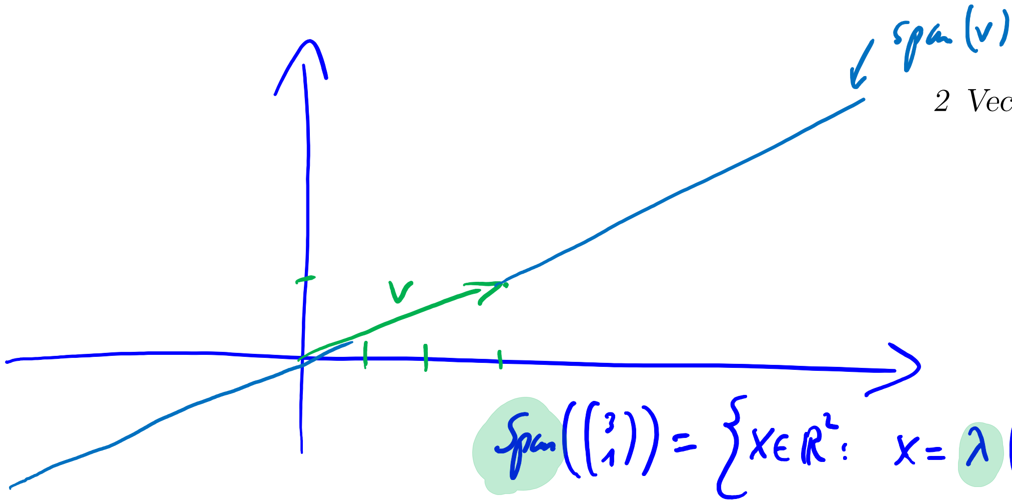
this is not a linear subspace in general



Example 2.17. Let us look at some spans:

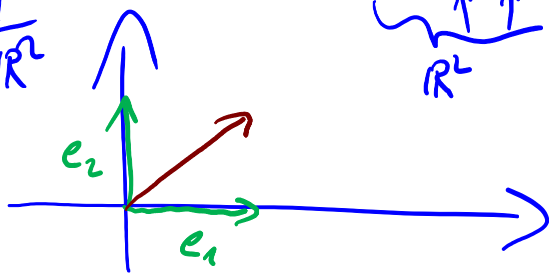
(a) $\operatorname{Span}(\binom{3}{1}) \subset \mathbb{R}^2$ is the line that "the vector $\binom{3}{1}$ spans" going trough the origin of $\mathbb{R}^2$.

" v



$\mathrm{Span}\left(\begin{pmatrix}3\\1\end{pmatrix}\right) = \left\{x \in \mathbb{R}^2 : x = \lambda \begin{pmatrix}3\\1\end{pmatrix} \text{ for a } \lambda \in \mathbb{R}\right\}$

(b) $\underbrace{\mathrm{Span}(\binom{1}{0}, \binom{0}{1})}_{\mathbb{R}^2}$ is the whole plane $\mathbb{R}^2$. $\underbrace{\mathrm{Span}(\binom{1}{0}, \binom{1}{1})}_{\mathbb{R}^2}$ is also the whole plane.



For $\begin{pmatrix}v_1\\v_2\end{pmatrix} \in \mathbb{R}^2$, we get

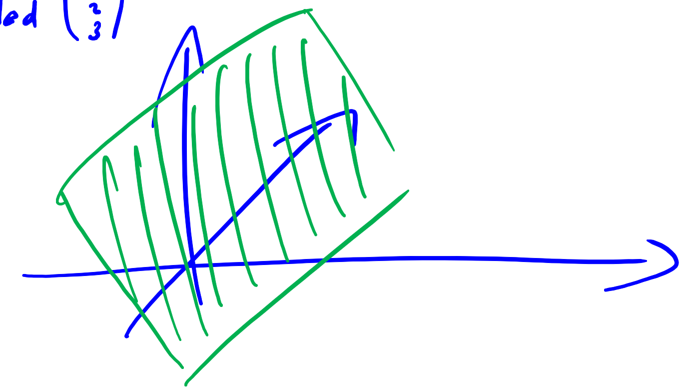
$$\begin{pmatrix}v_1\\v_2\end{pmatrix} = \underbrace{(v_1 - v_2)}_{\lambda_1}\begin{pmatrix}1\\0\end{pmatrix} + \underbrace{(v_2)}_{\lambda_2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}v_1\\v_2\end{pmatrix} \in \mathrm{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right).$$

(c) $\mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ this is the xy-plane in $\mathbb{R}^3$.

(d) $\mathrm{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\7\end{pmatrix}\right)$ is a plane in $\mathbb{R}^3$ going through $\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\7\end{pmatrix}$.

not a scaled $\begin{pmatrix}1\\2\\3\end{pmatrix}$



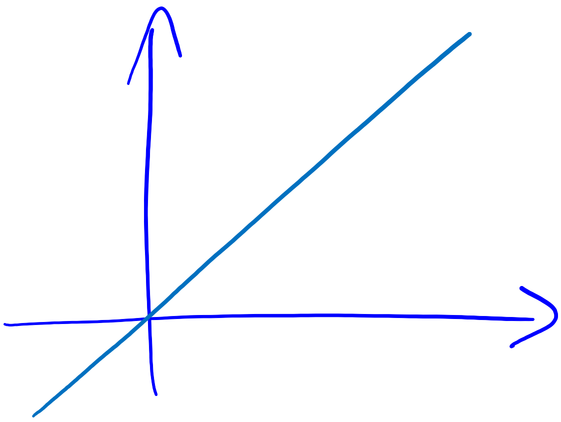
(e) $\underbrace{\mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)}_{\mathbb{R}^3}$ is the whole space $\mathbb{R}^3$. $\underbrace{\mathrm{Span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)}_{\mathbb{R}^3}$ is also the whole space

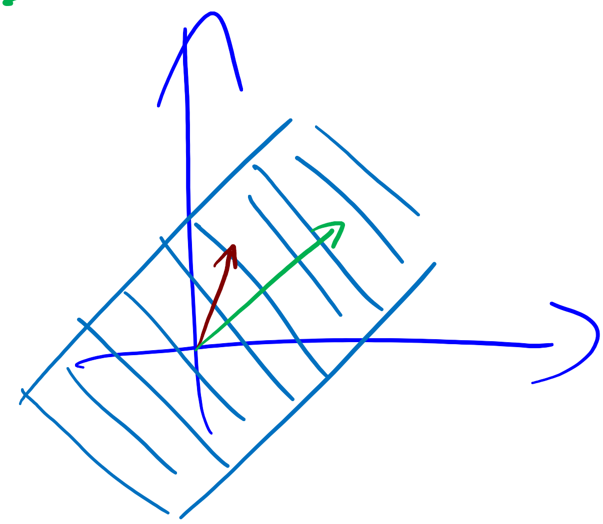
For $\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \in \mathbb{R}^3$, we have to find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \frac{1}{2}(+v_1 + v_2 - v_3)\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}(-v_1 + v_2 + v_3)\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}(+v_1 - v_2 + v_3)\begin{pmatrix}1\\0\\1\end{pmatrix}$$

(Solving linear equations $\rightsquigarrow$ Chapter 3)

(f) $\mathrm{Span}\left(\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\right)$ is a "line" in $\mathbb{R}^5$, $\mathrm{Span}\left(\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}5\\4\\3\\2\\1\end{pmatrix}\right)$ is a "plane".





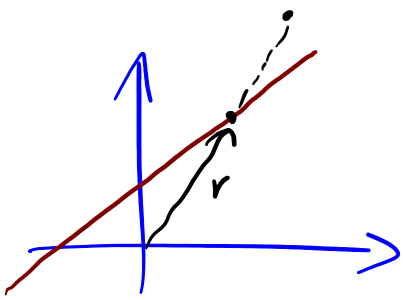
Affine subspaces and convex subsets

Rule of thumb:

Affine subspaces correspond to arbitrary lines, planes, . . . In other words: translated linear subspaces.

If we do not want $\mathbf{o}$ to be part of our "generalised plane", we have to replace linear combinations by *affine combinations*:

$$\mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \quad \text{where} \quad \sum_{j=1}^{k} \lambda_j = 1.$$

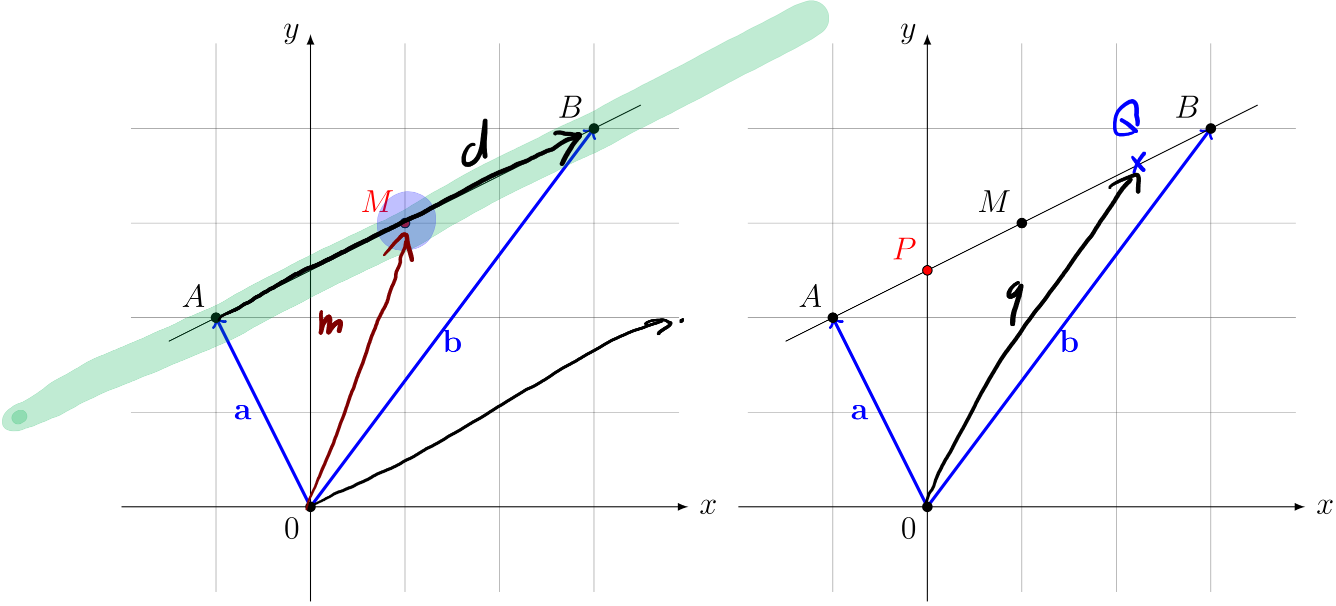


additional condition.

Example 2.18. Consider the position vectors

$$\mathbf{a} = \begin{pmatrix}-1\\2\end{pmatrix} \quad \text{und} \quad \mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix}$$

corresponding to the points A and B. Find the centre point of the line between A and B.



The connection vector from A to B is then:

$$-\mathbf{a}+\mathbf{b} = -\begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1+3\\-2+4\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix} =: \mathbf{d}$$

The center point is then given by going only half way in the direction of $\mathbf{d}$:

$$\mathbf{m} = \mathbf{a} + \frac{1}{2}\mathbf{d} = \begin{pmatrix}-1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \tag{2.6}$$

The point M with position vector $\mathbf{m} = \binom{1}{3}$ is the wanted centre point. In general, we get the formula:

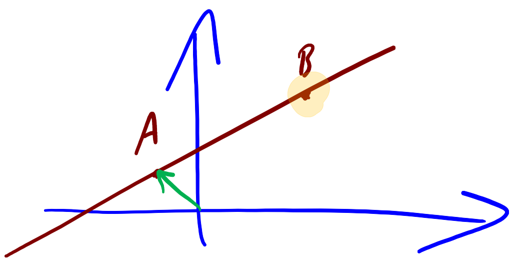
$$\mathbf{m} = \mathbf{a} + \tfrac{1}{2}\mathbf{d} = \mathbf{a} + \tfrac{1}{2}(-\mathbf{a}+\mathbf{b}) = \mathbf{a} - \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} = \underbrace{\tfrac{1}{2}}_{\lambda_1}\mathbf{a} + \underbrace{\tfrac{1}{2}}_{\lambda_2}\mathbf{b} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b})$$

linear combination
$\lambda_1 + \lambda_2 = 1$

Instead of using $\frac{1}{2}$, we can choose $\lambda \in \mathbb{R}$ to divide the line from A to B. We get:

$$\mathbf{q} := \mathbf{a} + \lambda\overbrace{(-\mathbf{a}+\mathbf{b})}^{\mathbf{d}} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} \tag{2.7}$$

linear combination with
$(1-\lambda) + \lambda = 1$

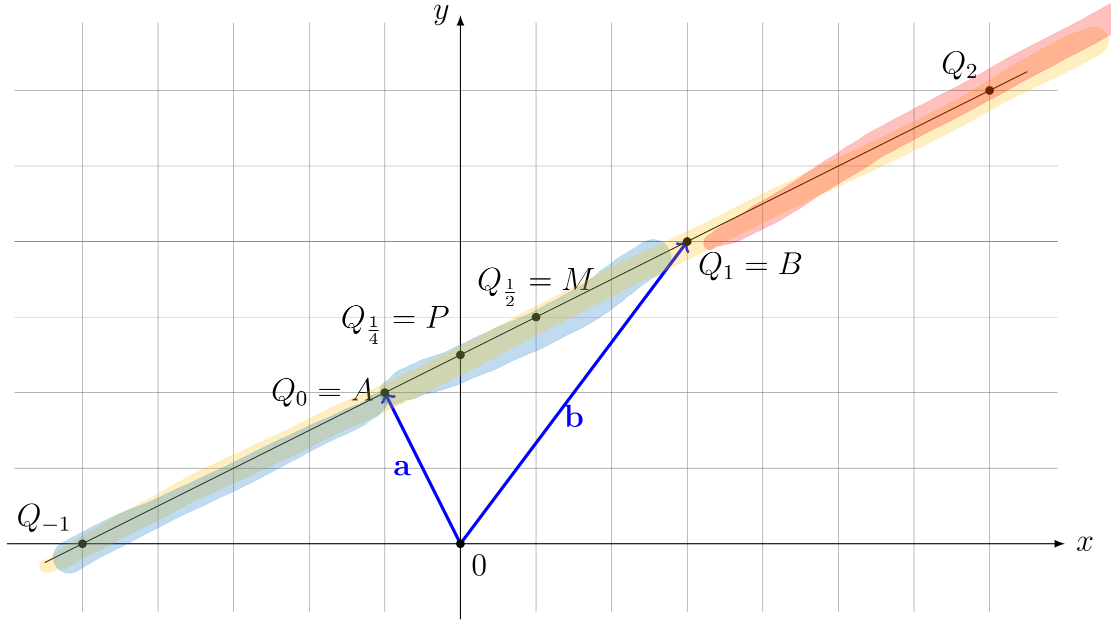


The corresponding point Q (with position vector $\mathbf{q}$ from the equation above) lies

at point A if $\lambda = 0$,
at point B if $\lambda = 1$,

$$\boxed{q = (1-\lambda)a + \lambda b}$$

$$\begin{aligned}
&\text{at the centre point } M && \text{if } \lambda = \tfrac{1}{2}, \\
&\text{between } A \text{ and } B && \text{if } \lambda \in [0,1] && (2.8)\\
&\text{on the line through } A \text{ and } B && \text{for all } \lambda \in \mathbb{R}, && (2.9)\\
&\text{on the line through } A \text{ and } B, \text{“in front of” } A && \text{for all } \lambda < 0,\\
&\text{on the line through } A \text{ and } B, \text{“behind” } B && \text{for all } \lambda > 1.
\end{aligned}$$

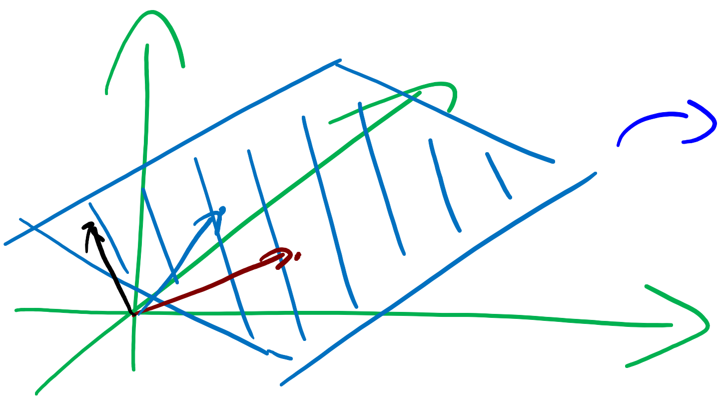


This brings out to the following:

Definition 2.19. Affine Subspaces in $\mathbb{R}^n$

A subset $U \subset \mathbb{R}^n$ is called an affine subspace of $\mathbb{R}^n$ if all affine combinations of vectors in U remain also in U:

$$\mathbf{u}_1, \ldots, \mathbf{u}_k \in U\,,\ \lambda_1, \ldots, \lambda_k \in \mathbb{R} \text{ with } \sum_{j=1}^{k} \lambda_j = 1 \implies \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \in U$$



→ If $0 \notin U$, then U is not a linear subspace.

Definition 2.20. Convex subsets in $\mathbb{R}^n$

A subset $U \subset \mathbb{R}^n$ is called a convex subset *of $\mathbb{R}^n$ if all* convex combinations *of vectors in U remain also in U:*

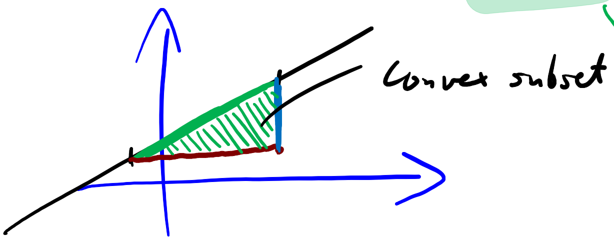
$$\mathbf{u}_1, \dots, \mathbf{u}_k \in U, \ \lambda_1, \dots, \lambda_k \in [0,1] \text{ with } \sum_{j=1}^{k} \lambda_j = 1 \implies \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \in U$$

The analogous formulation to the linear hull is the *affine hull*. Try to give a definition!

span

$\text{affspan}(M) := \{ \dots \}$

$\text{Conv span}(M) := \{ \dots \}$ affine



$$\sum_{j=1}^{k} \lambda_j u_k = \sum_{j=1}^{k+1} \lambda_j u_k$$

$u_{k+1} := 0$

$\lambda_{k+1} = 1 - \sum_{j=1}^{k} \lambda_j$

Proposition 2.21. Properties of affine subspaces

(i) Each linear subspace is an affine subspace. (by definition) ✓

(ii) If an affine subspace contains $\mathbf{o}$, it is a linear subspace.

(iii) Given an affine subspace $S \subset \mathbb{R}^n$ and a vector $\mathbf{v} \in \mathbb{R}^n$, the translated set:

$$\mathbf{v} + S := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{v} + \mathbf{s} \text{ for } \mathbf{s} \in S\}$$

is also an affine subspace.

(iv) Every nonempty affine subspace S can be written in the form $S = \mathbf{v} + U$ for some $\mathbf{v} \in S$ and U a linear subspace.

Proof. (i) : Follows from the definition because each affine combination is a linear combination.

(ii) : If we have an arbitrary linear combination, we can trivially add also the zero vector. But if the zero-vector is in a linear combination, we can make it an affine one.

(iii) : Let us write an arbitrary affine combination of elements of $\mathbf{v} + S$: $\exists\, w_j$ and $\lambda_j \in \mathbb{R}$ with $\sum_{j=1}^{k} \lambda_j = 1$

$w_j = v + s_j$

$$\sum_{j=1}^{k} \lambda_j w_j = \sum_{j=1}^{k} \lambda_j \underbrace{(\mathbf{s}_j + \mathbf{v})}_{\in \mathbf{v}+S} = \sum_{j=1}^{k} \lambda_j \mathbf{s}_j + \underbrace{\left(\sum_{j=1}^{k} \lambda_j\right)}_{1} \mathbf{v} = \underbrace{\sum_{j=1}^{k} \lambda_j \mathbf{s}_j + \mathbf{v}}_{\in \mathbf{v}+S}.$$

affine combination

$\in S$

$\Longrightarrow$ $v+S$ is an affine subspace.

(iv) For S', choose $v \in S$. Then by (iii) $(-v) + S'$ is an affine subspace, and it contains o.

By (ii) $U := (-v) + S$ is a linear subspace. $\Longrightarrow S = v + U$ $\square$

Proposition 2.22. Characterisation of affine subspaces

Let $S \subset \mathbb{R}^n$, such that

$$\mathbf{a}, \mathbf{b} \in S\,,\ \lambda \in \mathbb{R} \implies \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \in S$$

Then S is already an affine subspace.

Proof. We do a proof by mathematical induction:

Induction hypothesis: affine combinations of k vectors remain in S. In other words:

$$\mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{a}_j \text{ and } \sum_{j=1}^{k} \lambda_j = 1 \text{ implies } \mathbf{v} \in S$$

for every k and every admissible choice of λ_j and $\mathbf{a}_j \in S$.

Base case: by assumption, this is certainly true for $k = 2$.

Induction step: $k \to k+1$. Let $\mathbf{a}_j$ and λ_j be given for all $j \in \{1, \ldots, k+1\}$. By definition

$$\lambda_1 + \cdots + \lambda_k = 1 - \lambda_{k+1}$$

thus we can write:

$$\mathbf{v} = \sum_{j=1}^{k+1} \lambda_j \mathbf{a}_j = (1-\lambda_{k+1}) \underbrace{\left(\sum_{j=1}^{k} \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_k} \mathbf{a}_j\right)}_{\text{affine combination } \mathbf{w}} + \lambda_{k+1}\mathbf{a}_{k+1}$$

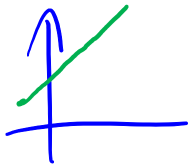
$$= (1-\lambda_{k+1})\mathbf{w} + \lambda_{k+1}\mathbf{u}_{k+1} \in S$$

By our induction hypothesis, $\mathbf{w} \in U$, because it is an affine combination of k vectors. Thus, $\mathbf{v} \in U$ as well, because it is an affine combination of $\mathbf{w}$ and $\mathbf{u}_{k+1}$. □

Conical combinations (an outlook)

There are also other rules for combining vectors. they lead to different classes of sets. For example, *conical combinations* of vectors are defined as:

$$\mathbf{v} = \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \text{ where } \lambda_j \geq 0.$$



The sets which contain all possible conical combinations of their elements are called *convex cones*, and we can define the conical hull of a set of vectors.

We can summarise this in the following table:

	no sign imposed	$\lambda_j \geq 0$
no sum imposed	linear	conical
$\sum \lambda_j = 1$	affine	convex

For all these types of sets we know ”... combinations“, and ”... hulls“.

This illustrates our strategy: describe things known from $\mathbb{R}^2$ and $\mathbb{R}^3$ algebraically, and thus generalise them to arbitrary dimensions.

2.5 Inner product and norm in $\mathbb{R}^n$

↳ more structure to our vector space

We transfer the notion of the inner product to define orthogonality and the length of the vector to the general $\mathbb{R}^n$

Definition 2.23. Inner product: $\langle$Vector, Vector$\rangle$ = Number

For two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \quad \text{the number} \quad \underbrace{\langle \mathbf{u}, \mathbf{v} \rangle}_{\in \mathbb{R}} := \underline{u_1 v_1 + \ldots + u_n v_n} = \sum_{i=1}^{n} u_i v_i$$

is called the (standard) inner product of $\mathbf{u}$ *and* $\mathbf{v}$*. Sometimes also called: (standard) scalar product. If* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$*, then we call* $\mathbf{u}$ *and* $\mathbf{v}$ *orthogonal.*

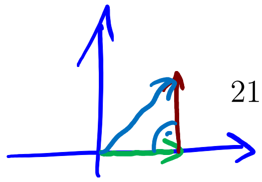
Proposition 2.24.

The standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *fulfils the following: For all vectors* $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$*, one has*

- **(S1)** $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ *for all* $\mathbf{x} \neq \mathbf{o}$, $x=0 \iff \langle x,x\rangle = 0$ *(positive definite)*
- **(S2)** $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$, *(additive)* } *(linear)*
- **(S3)** $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, *(homogeneous)* } *(linear)*
- **(S4)** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$. *(symmetric)*

↳ You need (S1)-(S4) to define non-standard inner products in $\mathbb{R}^n$.
↳ (Chapter 5)

$\langle x, y \rangle$ - measures angles (for example 90°)
- " lengths



Definition 2.25. Norm of a vector in $\mathbb{R}^n$

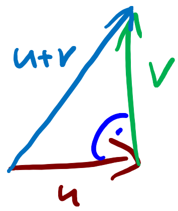
For a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{the number} \qquad \|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + \ldots + v_n^2}$$

is called the norm or length of $\mathbf{v}$.

$$\langle u+v, u+v\rangle \overset{(S2)}{=} \langle u, u+v\rangle + \langle v, u+v\rangle \overset{(S2),(S4)}{=} \langle u,u\rangle + \underbrace{\langle u,v\rangle}_{=0} + \underbrace{\langle v,u\rangle}_{=0} + \langle v,v\rangle$$

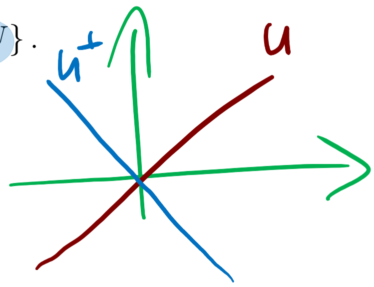
$$= \langle u,u\rangle + \langle v,v\rangle$$



$$\mathbf{u} \perp \mathbf{v} \quad \Rightarrow \quad \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

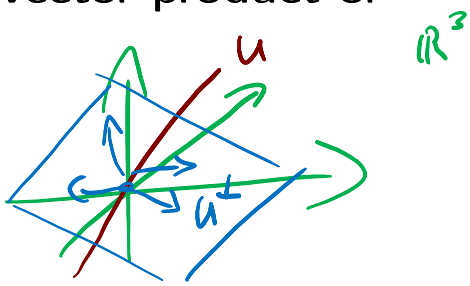
For a linear subspace $U \subset \mathbb{R}^n$ we define the orthogonal complement:

$$U^\perp := \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \quad \forall \mathbf{u} \in U\}\,.$$



However, we come back to such constructions later.

2.6 A special product in $\mathbb{R}^3(!)$: The vector product or cross product



Definition 2.26. Cross product: Vector $\times$ Vector $=$ Vector

The cross product or vector product of two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \quad \text{is given by} \quad \mathbf{u} \times \mathbf{v} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \in \mathbb{R}^3.$$