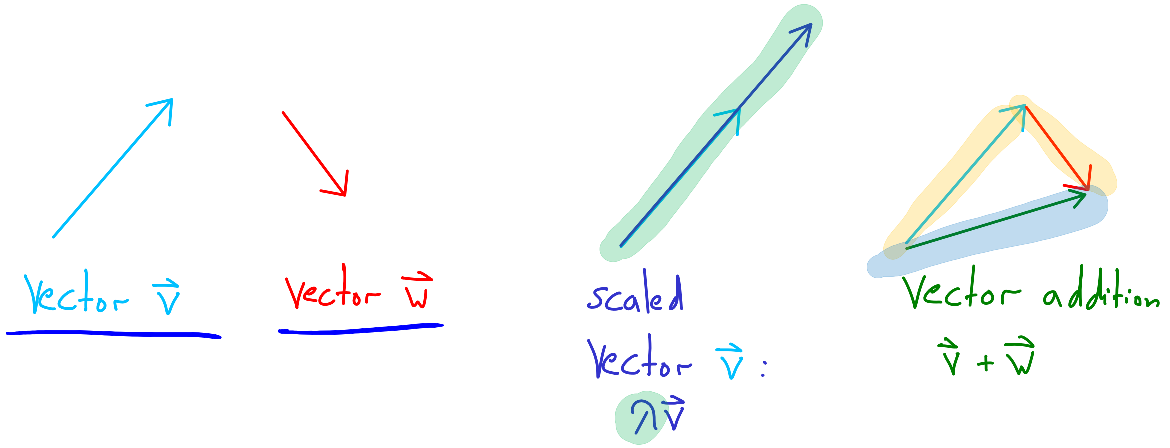
Knowledge:
- logic, sets
- Maps between sets (inj., surj., bij.)
  $f \circ g$
- induction (proof for all natural numbers)

2 Vectors in $\mathbb{R}^n$

vector space

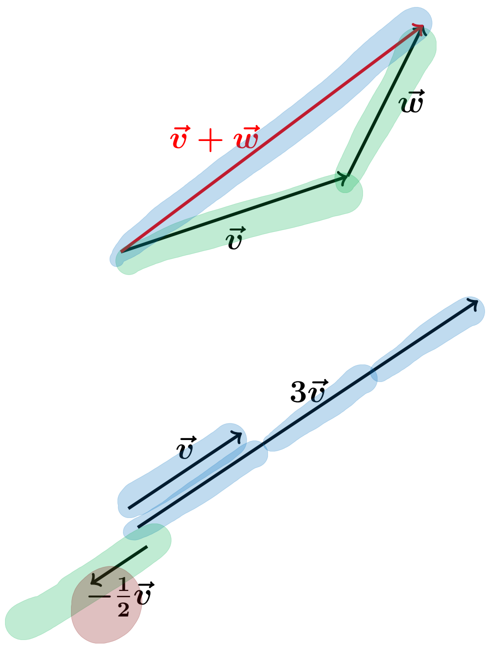
2.1 What are vectors?

In this section we do some informal discussions about the objects of linear algebra. We will make all objects into rigorous definitions later.



With vectors or arrows, you can do two things:

- Add *the two arrows, by concatenating them and call the result* $\vec{v}+\vec{w}$.
- Scale *an arrow by a (positive or negative) factor* λ *and call the result* $\lambda\vec{v}$.



"changes direction"

"doesn't change direction"

With these operations we can combine $\vec{v}$ and $\vec{w}$ to a large number of arrows and this is what one calls a *linear combination*:

> *If we scale two vectors $\vec{v}$ and $\vec{w}$ and add them, we get a new vector:*
>
> $$\lambda\vec{v} + \mu\vec{w} \qquad \textit{(linear combination)}$$

Mostly, there is no confusion which variables are vectors and which one are just numbers such that we will omit the arrow from now on. However, we will use **bold letters** in this script to denote vectors most of the time.

v

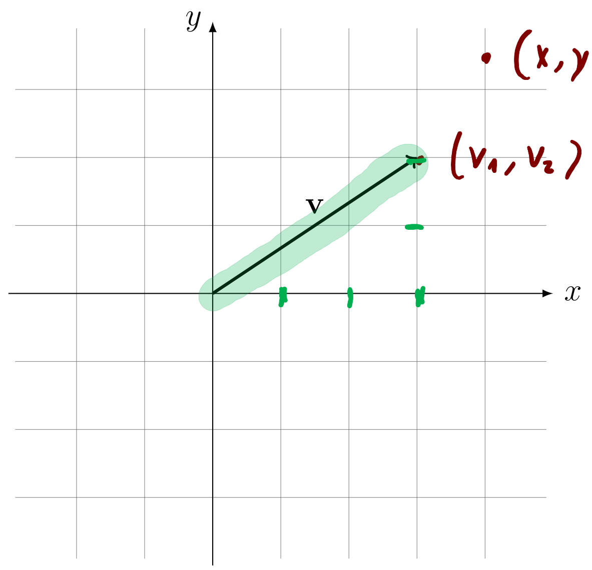
2.2 Vectors in the plane

We already know that we can describe the two-dimensional plane by the cartesian product $\mathbb{R} \times \mathbb{R}$, which consists of all the pairs of real numbers. For each point in the plane, there is an arrow where the tail sits at the origin. This is what one calls a *position vector*.

// $\mathbb{R}^2$

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

calculate in vector space



• $(x, y) \in \mathbb{R}^2$

$(v_1, v_2) = (3, 2)$

3 steps to the right
2 steps to the top

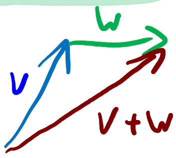
$v, w \in \mathbb{R}^2$

Now we also know how to add and scale these column vectors:

Define addition and scaling:

$$\mathbf{v}+\mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \qquad \lambda\mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

geometrical view:



numerical view:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Definition 2.1. Vector space $\mathbb{R}^2$

The set $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ is called the vector space $\mathbb{R}^2$ *if we write the elements in column form* → set $\mathbb{R}^2$ with $+, \cdot$

set $\in$

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{with} \qquad v_1, v_2 \in \mathbb{R}$$

and use the vector addition *and* scaling *from above. The numbers v_1 and v_2 are called the* components *of* $\mathbf{v}$.

For describing each point in the plane, the following elements are useful:

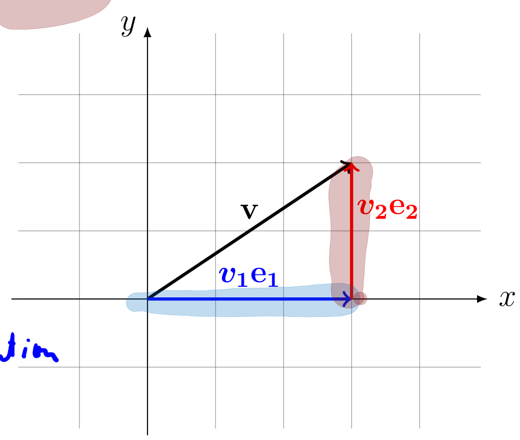
Definition 2.2. Zero vector and canonical unit vectors

The two vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^n$ are called canonical unit vectors *and* $\mathbf{o}$ *is called the* zero vector*:*

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that we can write every vector $\mathbf{v} \in \mathbb{R}^2$ as a linear combination of the two unit vectors:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{v_1\mathbf{e}_1 + v_2\mathbf{e}_2}_{\text{linear combination}}$$



Linear combinations

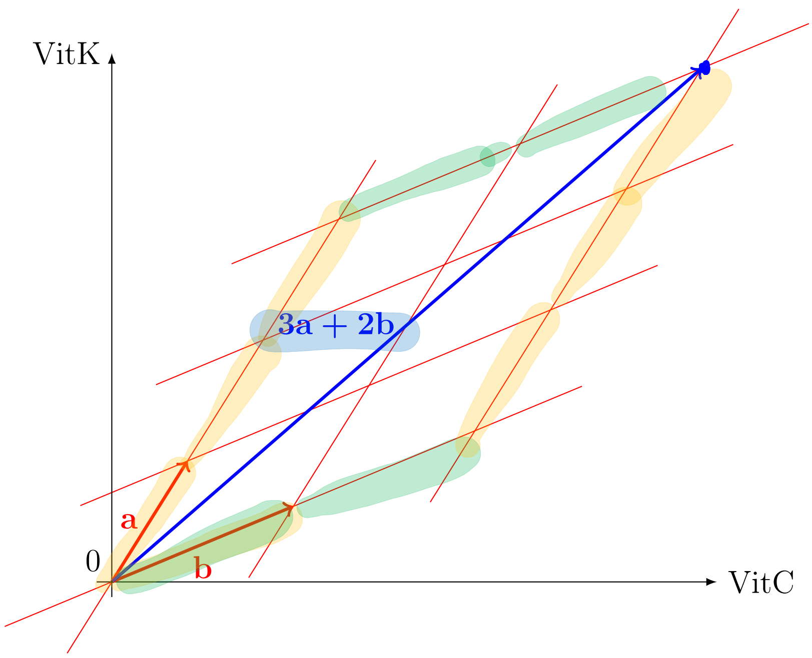
To compare apples and oranges: *An apple has 8mg vitamin C and 4µg vitamin K. An orange has 85mg vitamin C and 0.5µg vitamin K:*

$$\textit{Apple} \quad \mathbf{a} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{matrix}\scriptstyle\text{VitC} \\ \scriptstyle\text{VitK}\end{matrix}, \qquad \textit{Orange} \quad \mathbf{b} = \begin{pmatrix} 85 \\ 0.5 \end{pmatrix}\begin{matrix}\scriptstyle\text{VitC} \\ \scriptstyle\text{VitK}\end{matrix}$$

Fruit salad: *How much vitamin C and vitamin K do I get if I eat 3 apples and 2 oranges? Answer:*

$$3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} 8 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 85 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 3 \cdot 8 + 2 \cdot 85 \\ 3 \cdot 4 + 2 \cdot 0.5 \end{pmatrix} = \begin{pmatrix} 194 \\ 13 \end{pmatrix}\begin{matrix}\scriptstyle\text{VitC} \\ \scriptstyle\text{VitK}\end{matrix}$$

Here, you can see a rough sketch of this vector addition:



A vector written as

$$\lambda \mathbf{a} + \mu \mathbf{b} \quad \text{with} \quad \lambda, \mu \in \mathbb{R} \tag{2.1}$$

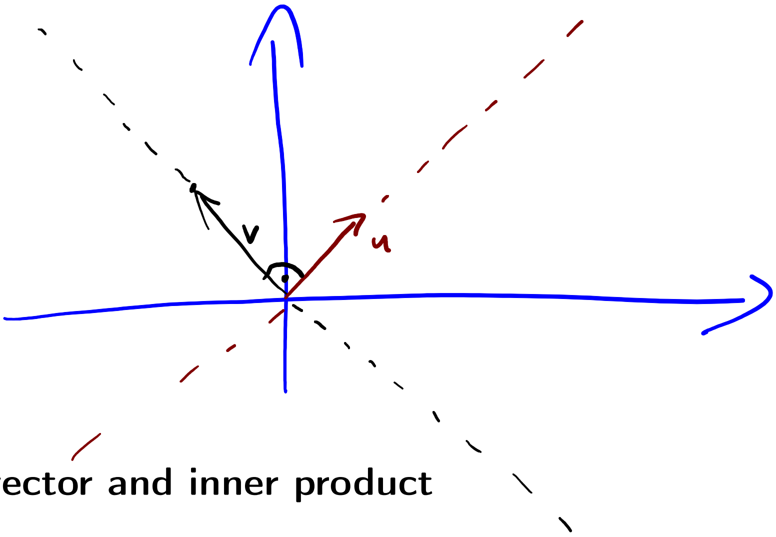
is called a *linear combination* of $\mathbf{a}$ and $\mathbf{b}$. We can expand this definition:

Definition 2.3. Linear combination

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^2$ *and* $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ *scalars. Then*

$$\sum_{j=1}^{k} \lambda_j \mathbf{v}_j = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

is called a linear combination of the vectors.

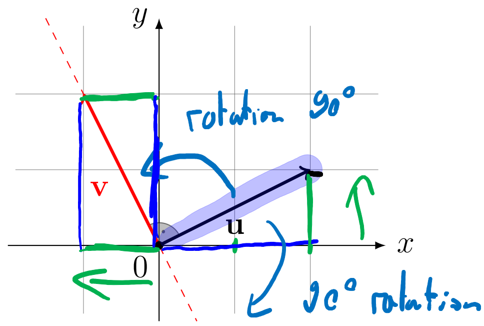


Orthogonal vector and inner product

Question:

Which vectors $\mathbf{v}$ *in* $\mathbb{R}^2$ *are perpendicular to the vector* $\mathbf{u} = \binom{2}{1}$*?*

Doing the sketch, one easily recognises that, for example, $\mathbf{v} = \binom{-1}{2}$ is perpendicular to $\mathbf{u}$. Of course, all multiples of this vector will also work. In general:



$$\mathbf{v} \in \mathbb{R}^2 \text{ is perpendicular to } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Longleftrightarrow \quad \mathbf{v} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \text{ for a } \lambda \in \mathbb{R} \tag{2.2}$$

Rule of thumb: orthogonal vector in $\mathbb{R}^2$

To find a vector that is orthogonal to $\binom{x}{y}$*, exchange the* x *and* y *and write a minus sign in front of* ***one*** *of the two.*

Looking at (2.2), we can reformulate:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ are orthogonal} \quad \Leftrightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \text{ for a } \lambda \in \mathbb{R}$$

$\Leftrightarrow$ $v_1 = -\lambda u_2$ and $v_2 = \lambda u_1$ "

$\Leftrightarrow$ $u_1 v_1 = (\underbrace{-\lambda u_1}_{= v_2}) u_2$ or $u_2 v_2 = (\underbrace{\lambda u_2}_{= -v_1}) u_1$

$$\Leftrightarrow \quad u_1 v_1 = -u_2 v_2$$
$$\Leftrightarrow \quad u_1 v_1 + u_2 v_2 = 0$$

Hence, this means that $\binom{u_1}{u_2}$ and $\binom{v_1}{v_2}$ are orthogonal if the calculation of $u_1 v_1 + u_2 v_2$ gives us 0. Therefore, the term $u_1 v_1 + u_2 v_2$ is used to define the so-called inner product or scalar product.

is related to an angle

Definition 2.4. Inner product: $\langle$vector, vector$\rangle$ = number

For two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \qquad \text{the number} \qquad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + u_2 v_2 = \sum_{i=1}^{2} u_i v_i$$

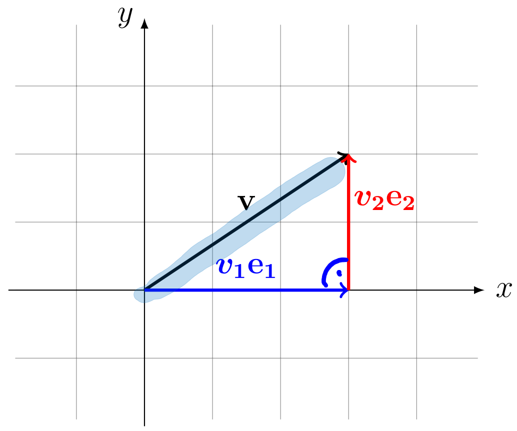
is called the (standard) inner product of $\mathbf{u}$ *and* $\mathbf{v}$*. Sometimes also called: (standard) scalar product.*

Definition 2.5. Orthogonality of two vectors in $\mathbb{R}^2$

Two vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *are called orthogonal (or perpendicular) if* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ *holds. We also denote this by* $\mathbf{u} \perp \mathbf{v}$

By using Pythagoras' theorem, we can calculate the length of the arrow in the coordinate system.

Length of $\mathbf{v} = \sqrt{v_1^2 + v_2^2}$



Obviously, we can also define it by using the inner product:

Definition 2.6. Norm of a vector in $\mathbb{R}^2$

For a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \qquad \text{the number} \qquad \|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2}$$

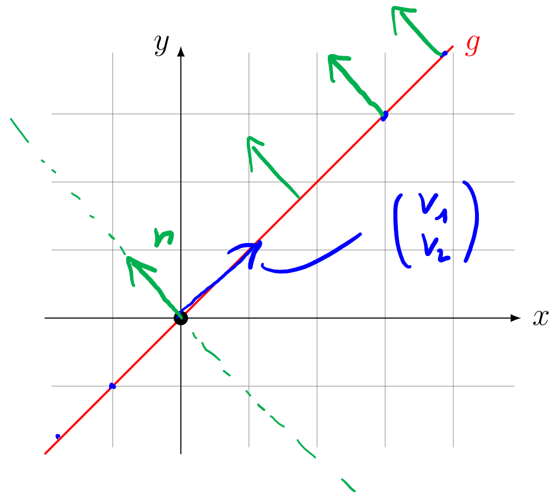
same vector

is called the norm or length of $\mathbf{v}$.

Lines in $\mathbb{R}^2$

For describing points in the plane, we can use the position vectors and just use the vector operations to define objects in the plane. One of the simplest objects is a line g inside the plane:

First case: The origin lies on the line g.



$$g = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ for } \lambda \in \mathbb{R} \right\}$$

line g $\leftrightsquigarrow$ a normal vector n (orthogonal to g)

$g = \{$ position vectors that are orthogonal to $n \}$

$$n = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

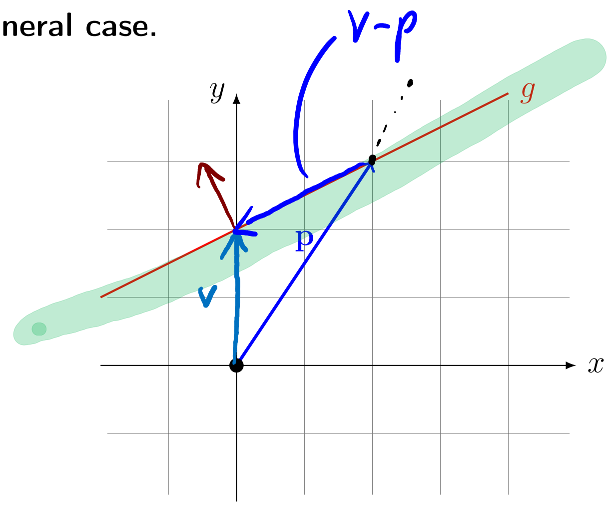
$$0 = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \alpha \cdot x + \beta \cdot y$$

$$g = \{ \mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{n}, \mathbf{v} \rangle = 0 \} = \{ \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{v}} \in \mathbb{R}^2 : \underbrace{\alpha x + \beta y}_{\langle \mathbf{n}, \mathbf{v} \rangle} = 0 \}.$$

Example: $g = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 3x \right\} \iff 3x - y = 0$

$$n = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Second case: General case.



$n = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$

$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

v lies on g $\iff \langle n, v-p \rangle = 0$

$\iff \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\rangle = 0$

$\iff \alpha(x-p_1) + \beta(y-p_2) = 0$

$\iff \alpha x + \beta y = \alpha p_1 + \beta p_2 =: \delta$

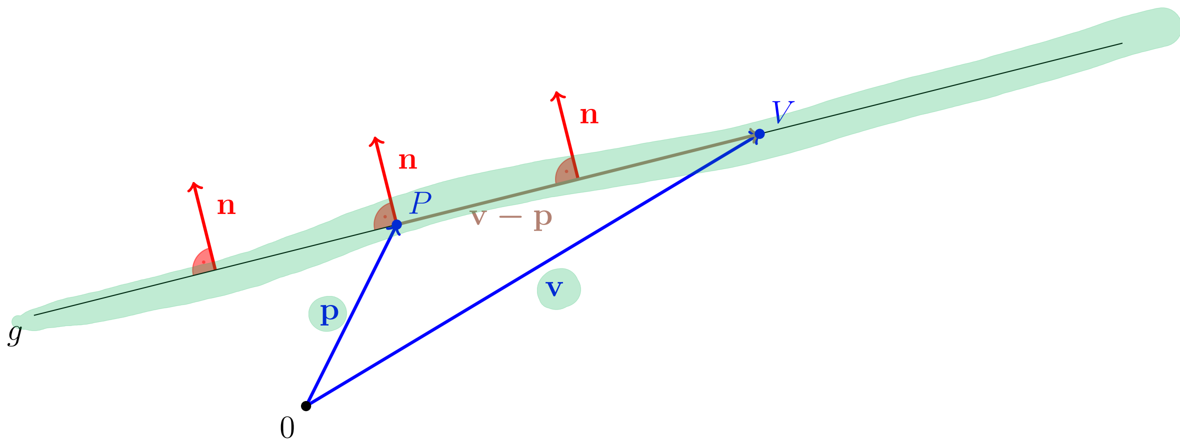
Lines in the plane $\mathbb{R}^2$ (Equation in normal form)

For each line g, one has the following representation:

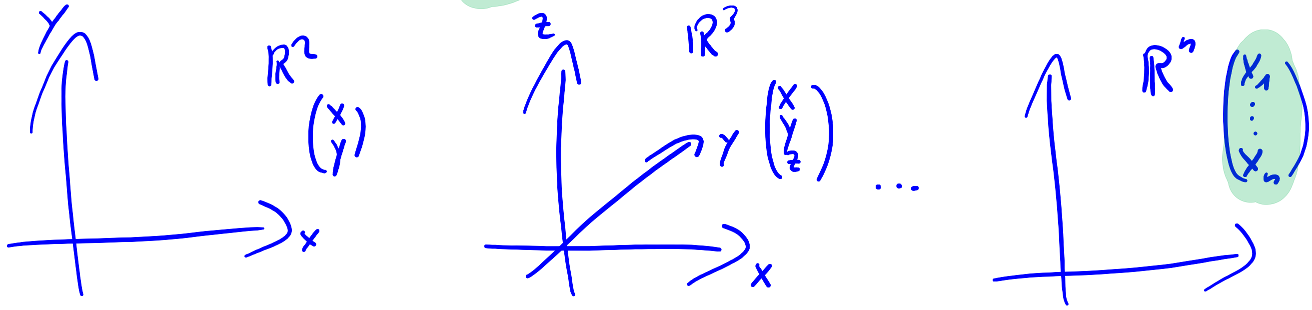
$$g = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\} = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \alpha x + \beta y = \delta\}$$

with $\delta := \alpha p_1 + \beta p_2 = \langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rangle = \langle \mathbf{n}, \mathbf{p} \rangle$. *If the origin lies on g, then $\delta = 0$ (choose $\mathbf{p} = \mathbf{o}$).*

Example: $g = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 3x + 6 \right\}$ $\quad n = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \ \delta = -6$



2.3 The vector space $\mathbb{R}^n$



$$\lambda \in \mathbb{R}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \lambda\mathbf{v} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$(x_1, \dots, x_n)$

in $\mathbb{R}$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

Definition 2.7. Vector space $\mathbb{R}^n$

The set $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is called the vector space $\mathbb{R}^n$ *if we write the elements in column form*

with +, ·

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \textit{with} \qquad v_1, \dots, v_n \in \mathbb{R}$$

and use the vector addition and scaling from above. The number v_i are called the ith component *of $\mathbf{v}$.*

$(\mathbb{R}^n, +)$ commutative group

Proposition 2.8. Properties of the vector space $\mathbb{R}^n$

The set $V = \mathbb{R}^n$ with the addition $+$ and scalar multiplication $\cdot$ fulfils the following:

(1) $\forall \mathbf{v}, \mathbf{w} \in V: \quad \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ *(+ is commutative)*

(2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V: \quad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *(+ is associative)*

(3) *There is a zero vector $\mathbf{o} \in V$ with the property: $\forall \mathbf{v} \in V$ gilt: $\mathbf{v} + \mathbf{o} = \mathbf{v}$.*

(4) *For all $\mathbf{v} \in V$ there is a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.*

(5) *For the number $1 \in \mathbb{R}$ and each $\mathbf{v} \in V$, one has: $1 \cdot \mathbf{v} = \mathbf{v}$.*

(6) $\forall \alpha, \beta \in \mathbb{R} \quad \forall \mathbf{v} \in V: \quad \lambda \cdot (\mu \cdot \mathbf{v}) = (\lambda\mu) \cdot \mathbf{v}$ *($\cdot$ is "associative")*

(7) $\forall \lambda \in \mathbb{R} \quad \forall \mathbf{v}, \mathbf{w} \in V: \quad \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w})$ *(distributive $\cdot +$)*

(8) $\forall \lambda, \mu \in \mathbb{R} \quad \forall \mathbf{v} \in V: \quad (\lambda + \mu) \cdot \mathbf{v} = (\lambda \cdot \mathbf{v}) + (\mu \cdot \mathbf{v})$ *(distributive $+ \cdot$)*

Each set V with an addition and scalar multiplication that satisfies the eight rules above is called a vector space. We will come back to this in an abstract sense later. First we will use this notion to talk about vector spaces inside $\mathbb{R}^n$.

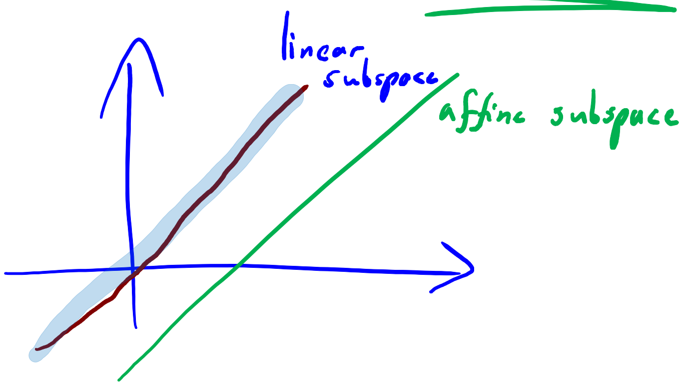
Definition 2.9. Zero vector and canonical unit vectors

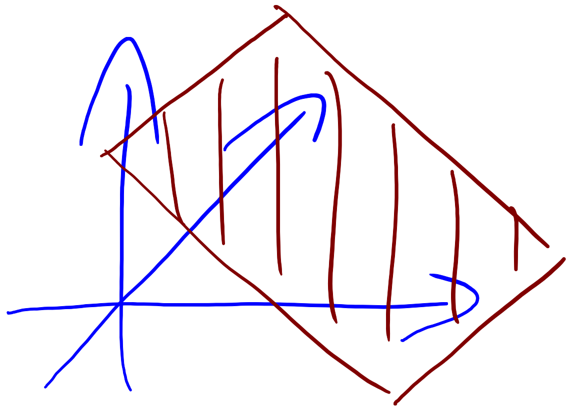
For $i = 1, \dots, n$, we denote the ith canonical unit vector by $\mathbf{e}_i \in \mathbb{R}^n$ and the zero vector by $\mathbf{o} \in \mathbb{R}^n$, which means:

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \mathbf{e}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$v \in \mathbb{R}^n: \qquad v = v_1 \cdot e_1 + \dots + v_n \cdot e_n = \sum_{i=1}^{n} v_i e_i$$

2.4 Linear and affine subspaces (and the like)





Linear subspaces

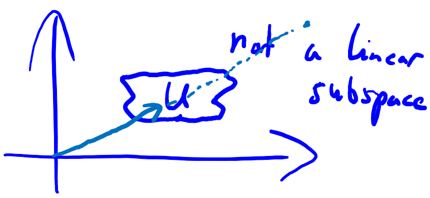
Rule of thumb:

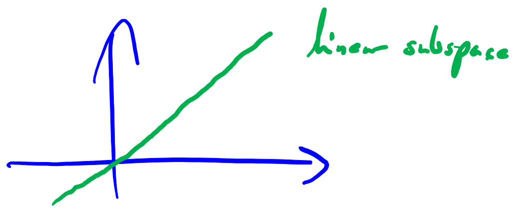
Linear subspaces correspond to lines, planes,... through the origin.

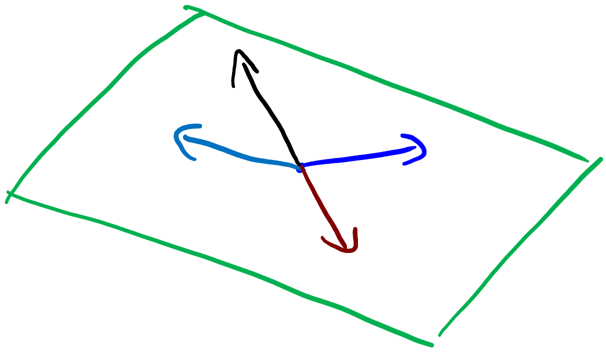
Definition 2.10. Subspaces in $\mathbb{R}^n$

A (nonempty) subset $U \subset \mathbb{R}^n$ is called a (linear) subspace of $\mathbb{R}^n$ if all linear combinations of vectors in U remain also in U:

$$\mathbf{u}_1, \dots, \mathbf{u}_k \in U, \lambda_1, \dots, \lambda_k \in \mathbb{R} \implies \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \in U.$$







Each linear subspace U of the vector space $\mathbb{R}^n$ is also a vector space in the sense of the properties given in Proposition 2.8.

Linear combinations remain in U (by definition), and rules are inherited from V.

Proposition 2.11. Characterisation for subspaces

Let $U \subset \mathbb{R}^n$, such that

$$\mathbf{u}, \mathbf{v} \in U, \ \lambda, \mu \in \mathbb{R} \implies \lambda\mathbf{u} + \mu\mathbf{v} \in U\,. \tag{2.3}$$

Then U is already a linear subspace.

Proof. We do the proof by induction for k vectors like in the definition of a subspace:

Induction hypothesis **(IH)**: Linear combinations of k vectors remain in U.

Base case **(BC)**: For $k = 2$. This is exactly given by equation (2.3).

Induction step **(IS)**: $k \to k+1$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1} \in U$ and $\lambda_1, \ldots, \lambda_{k+1}$ be given. We can write:

$$\mathbf{v} := \sum_{j=1}^{k+1} \lambda_j \mathbf{u}_j = \underbrace{\left(\sum_{j=1}^{k} \lambda_j \mathbf{u}_j \right)}_{=:\mathbf{w}} + \lambda_{k+1}\mathbf{u}_{k+1}$$
$$= \mathbf{w} + \lambda_{k+1}\mathbf{u}_{k+1} \in U$$

By our induction hypothesis, $\mathbf{w} \in U$ because it is a linear combination of k vectors. Thus, $\mathbf{v} \in U$ as well because it is a linear combination of $\mathbf{w}$ and $\mathbf{u}_{k+1}$, see (2.3). □

Examples: trivial subspaces: $\{0\}$, $\mathbb{R}^n$

All other subspaces: U $\quad \{0\} \subset U \subset \mathbb{R}^n$

Prove $g = \{\binom{x}{y} : x+y=0\}$ is a subspace

Choose $v, w \in g$, $\lambda_1, \lambda_2 \in \mathbb{R}$

...>

To prove that U is a linear subspace:

Check if two vectors linearly combined stay in U.

Rule of thumb: Subspace

A set U is a subspace if, by applying the operations $+$ and $\lambda\cdot$ on elements of U, one cannot escape the set U.

$U = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is not a subspace?

$2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin U$!

Linear hull or span

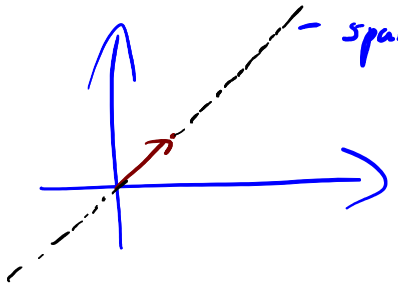
If we take a set of vectors $M \subset \mathbb{R}^n$, we can create a linear subspace by building all possible linear combinations:

Definition 2.12. Span

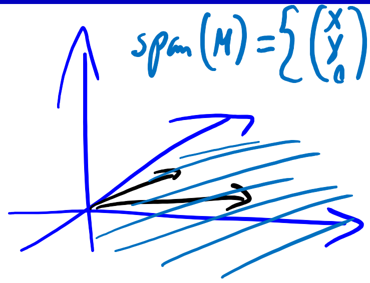
Let $M \subset \mathbb{R}^n$ be any non-empty subset. Then we define:

$$\operatorname{Span}(M) := \left\{ \mathbf{u} \in \mathbb{R}^n : \exists \lambda_j \in \mathbb{R}, \mathbf{u}_j \in M \text{ such that } \mathbf{u} = \sum_{j=1}^{k} \lambda_j \mathbf{u}_j \right\} .$$

This subspace is called the span or the linear hull of M. For convenience, we define $\operatorname{Span}(\emptyset) := \{\mathbf{o}\}$.



$-\ \operatorname{span}\left(\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}\right)$



$\operatorname{span}(M) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Rule of thumb: All linear combinations form the span

Every vector in $\operatorname{Span}(M)$ *can be written (possibly in several ways) as a linear combination of elements of M. Vice versa, every linear combination of M is contained in* $\operatorname{Span}(M)$.

$\operatorname{Span}(\{u_1, \dots, u_k\}) =: \operatorname{Span}(u_1, \dots, u_k)$, $\operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \text{Line}$

$U = \operatorname{Span}(u_1, \dots, u_k)$: U is generated by the vectors $u_1, \dots, u_k$

Example 2.13. The vector space $\mathbb{R}^n$ is spanned by the n unit vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{e}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

because $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$ for all $\mathbf{v} \in \mathbb{R}^n$. In short: $\mathbb{R}^n = \operatorname{Span}(\mathbf{e}_1, \dots, \mathbf{e}_n)$.