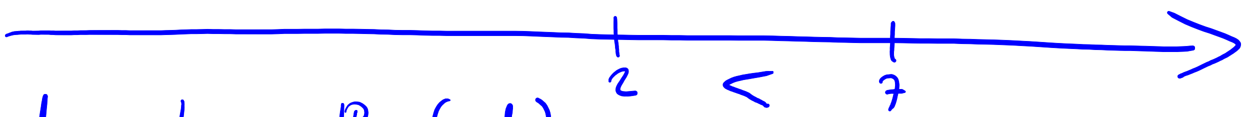
(c) $X_3 := \{|a-b| \colon a, b \in \{1,2,3\}\}$

(d) $X_4 := \{1, ..., 20\} \setminus \{n \in \mathbb{N} \colon \exists a, b \in \mathbb{N} \text{ with } 2 \le a \text{ and } 2 \le b \text{ and } n = a \cdot b\}$.

(e) $X_5 := \{S \colon S \subset \{1,2,3\}\}$.

VL1

1.2 Real Numbers



Real number:
- $\mathbb{R}$ (set)
- addition in $\mathbb{R}$ $+$ ($\rightsquigarrow$ subtraction)
- multiplication in $\mathbb{R}$ $\cdot$ ($\rightsquigarrow$ division)
- an ordering $<$

(vectors, matrices
⋮)

Some *laws* apply:

$$a + (b + c) = (a + b) + c, \qquad a(bc) = (ab)c \qquad \text{associative law}$$

$$a + b = b + a \qquad ab = ba \qquad \text{commutative law}$$

$$a(b + c) = ab + ac \qquad \text{distributive law}$$

Furthermore, we are used to have the neutral numbers 0 and 1 with special properties:

$$a + 0 = a \qquad a \cdot 1 = a$$

and additive inverse element $-a$ and also the multiplicative inverse $a^{-1} = 1/a$ for $a \neq 0$. They fulfil $a + (-a) = 0$ and $aa^{-1} = 1$.

A set with such properties is called a *field*. Here we have the field of real numbers $\mathbb{R}$.

Examples: $\mathbb{Q}$ field, $\mathbb{Z}_2 = \{0,1\}$ field

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	0

$1+1=1$
$a \in \mathbb{Z}_2$ $a \cdot 1 = a$
$\to 1\cdot(1+1) = 1+1 = 1$

(*)

(**)

Ordering: $<$

- For any $a \in \mathbb{R}$ exactly one of the three relations hold

$$a < 0, \text{ or } a > 0 \text{ or } a = 0$$

- For all $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ one has $a + b > 0$ and $ab > 0$.

Then, as a definition we write:

$$a < b \quad :\Leftrightarrow \quad a - b < 0$$

and

$$a \leq b \quad :\Leftrightarrow \quad a - b < 0 \text{ or } a = b\,.$$

Claim: If $x > 0$, then $-x < 0$. $(A \to B)$

Proof: Assume $x > 0$ and $-x \geq 0$. $(A \wedge \neg B) \Leftrightarrow (\neg(A \to B))$

1st case: $-x = 0$: $0 = x + (-x) = x \Rightarrow x = 0$ ↯ (contradiction)

2nd case: $-x > 0$: $0 \overset{\text{inverses}}{=} x + (-x) \overset{(**)}{>} 0 \Rightarrow 0 > 0$ ↯ (contradiction)

$\Rightarrow (A \Rightarrow B)$ $\square$

For describing subsets of the real numbers, we will use *intervals*. Let $a, b \in \mathbb{R}$. Then we define

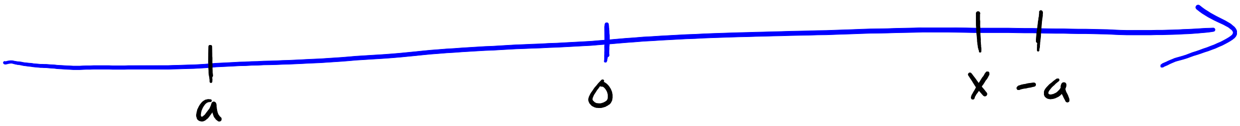
$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

If $a = b$, then $[a,b] = \{a\}$

Obviously, in the case $a > b$, all the sets above are empty. We also can define unbounded intervals:

∞ is just a symbol
is not a real number

$$[a, \infty) := \{x \in \mathbb{R} : a \leq x\}, \qquad (a, \infty) := \{x \in \mathbb{R} : a < x\}$$
$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}, \qquad (-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$



Definition 1.27. Absolute value for real numbers

The absolute value of a number $x \in \mathbb{R}$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

$|x| \in [0, \infty)$

Question 1.28. *Which of the following claims are true?*

$$|-3.14| = 3.14, \quad |3| = 3, \quad |-\tfrac{7}{5}| = \tfrac{7}{5}, \quad -|-\tfrac{3}{5}| = \tfrac{3}{5}, \quad |0| \text{ is not well-defined.}$$

Proposition 1.29. Two important properties

For any two real numbers $x, y \in \mathbb{R}$, one has

(a) $|x \cdot y| = |x| \cdot |y|$, *($|\cdot|$ is multiplicative),*

(b) $|x + y| \leq |x| + |y|$, *($|\cdot|$ fulfils the triangle inequality).*

Proof: Do it for own!

(∗) Supplementary details: Real numbers

The real numbers are a non-empty set $\mathbb{R}$ together with the operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and an ordering relation $< : \mathbb{R} \times \mathbb{R} \to \{\text{True}, \text{False}\}$ that fulfil the following rules

Q

(A) Addition

- (A1) associative: $x + (y + z) = (x + y) + z$
- (A2) neutral element: There is a (unique) element 0 with $x + 0 = x$ for all x.
- (A3) inverse element: For all x there is a (unique) y with $x + y = 0$. We write for this element simply $-x$.
- (A4) commutative: $x + y = y + x$

(M) Multiplication

- (M1) associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- (M2) neutral element: There is a (unique) element $1 \neq 0$ with $x \cdot 1 = x$ for all x.
- (M3) inverse element: For all $x \neq 0$ there is a (unique) y with $x \cdot y = 1$. We write for this element simply x^{-1}.
- (M4) commutative: $x \cdot y = y \cdot x$

(D) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

(O) Ordering

- (O1) for given x, y exactly one of the following three assertions is true: $x < y$, $y < x$, $x = y$.
- (O2) transitive: $x < y$ and $y < z$ imply $x < z$.
- (O3) $x < y$ implies $x + z < y + z$ for all z.
- (O4) $x < y$ implies $x \cdot z < y \cdot z$ for all $z > 0$.
- (O5) $x > 0$ and $\varepsilon > 0$ implies $x < \varepsilon + \cdots + \varepsilon$ for sufficiently many summands.

analysis!

(C) Completeness: Every sequence $(a_n)_{n \in \mathbb{N}}$ with the property *[For all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|a_n - a_m| < \varepsilon$ for all $n, m > N$]* has a limit.

(∗) Supplementary details: Definition: field

Every set M together with two the operations $+ : M \times M \to M$ and $\cdot : M \times M \to M$ that fulfil (A), (M) and (D) is called a *field*.

$\mathbb{Z}_2$

Sums and products

We will use the following notations.

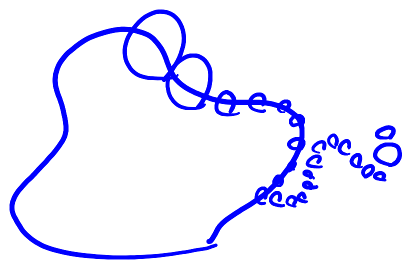
$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1} \cdot a_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n$$

The union works also for an arbitrary index set $\mathcal{I}$:

$$\bigcup_{i \in \mathcal{I}} A_i = \{x : \exists i \in \mathcal{I} \text{ with } x \in A_i\} .$$



The first is a useful notation for a *sum* which is the result of an addition. Two or more *summands* added. Instead of using points, we use the Greek letter $\sum$. For example,

$$3 + 7 + 15 + \ldots + 127$$

is not an unambiguous way to describe the sum. Using the sum symbol, there is no confusion:

$$\sum_{i=2}^{7}(2^i - 1).$$

Of course, the parentheses are necessary here. You can read this as a `for` loop:

for loop for the sum above

```
sum := 0;
for i:=2 to 7 do {
  sum := sum + (2^i-1);
}
```

Rule of thumb: Let i run from 2 to 7, calculate $2^i - 1$ and add.

index variable:	$i = 2$,	*first summand:*	$2^i - 1 = 2^2 - 1 =$	$4 - 1 =$	3
index variable:	$i = 3$,	*second summand:*	$2^i - 1 = 2^3 - 1 =$	$8 - 1 =$	7
index variable:	$i = 4$,	*third summand:*	$2^i - 1 = 2^4 - 1 =$	$16 - 1 =$	15
index variable:	$i = 5$,	*fourth summand:*	$2^i - 1 = 2^5 - 1 =$	$32 - 1 =$	31
index variable:	$i = 6$,	*fifth summand:*	$2^i - 1 = 2^6 - 1 =$	$64 - 1 =$	63
index variable:	$i = 7$,	*last summand:*	$2^i - 1 = 2^7 - 1 =$	$128 - 1 =$	127
		Sum:			246

Example 1.30.

$$\sum_{i=1}^{10}(2i - 1) = 1 + 3 + 5 + \ldots + 19 \stackrel{?}{=} 100$$

$$\sum_{i=-10}^{10} i = -10 - 9 - 8 - \ldots - 1 + 0 + 1 + \cdots + 8 + 9 + 10 \stackrel{?}{=} 0$$

With the same construction, we describe the result of a multiplication, called a *product*, which consists of two or more *factors*. There we use the Greek letter $\prod$. For example:

$$\prod_{i=1}^{8}(2i) = (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \ldots \cdot (2 \cdot 8) \stackrel{?}{=} 10321920.$$

Rational versus real numbers

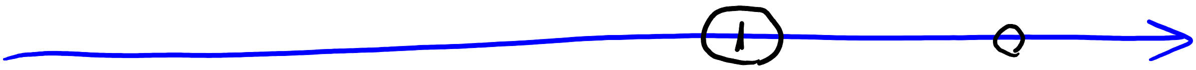
For most practical purposes the rational numbers (all fractions)

$$\mathbb{Q} = \left\{ x : x = \frac{n}{d} \text{ with } n \in \mathbb{Z},\ d \in \mathbb{N} \right\}$$

are enough. All numbers that can somehow be stored sensibly on a computer are rational.

- Each real number can be approximated by a rational number.
- irrational: $\sqrt{2}$, π, e, φ ← golden ratio

$|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$



Mathematicians say: $\mathbb{R}$ is complete, $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R}$ is the completion of $\mathbb{Q}$.
We come back to this in the lecture Mathematical Analysis.

1.3 Maps

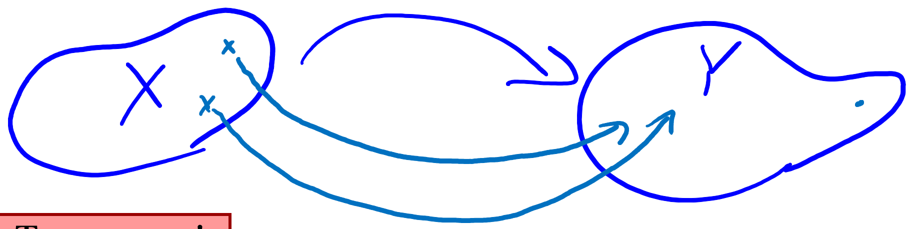
Definition 1.31. Function or map

Let X, Y be non-empty sets. A rule that assigns to each argument $x \in X$ a unique value $y \in Y$ is called a map or function from X into Y. One writes for this y usually $f(x)$.

Notation:

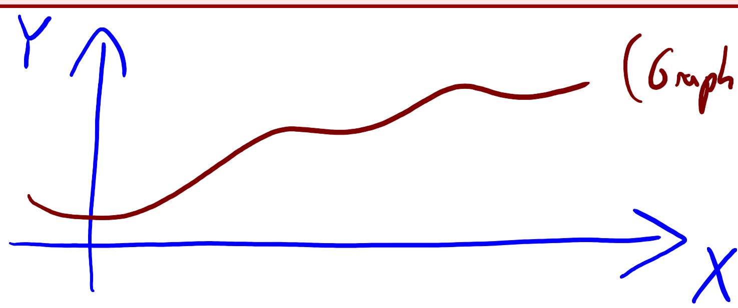
$$f : X \to Y$$
$$x \mapsto f(x)$$

Here, X is called domain of f, and Y is called codomain.



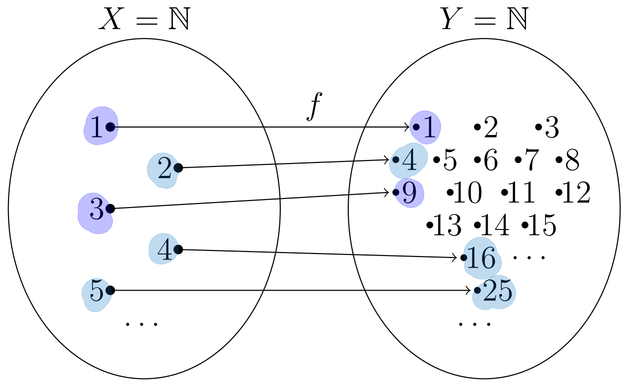
Attention! Two arrows!

We use the arrow "$\to$" only between the sets, domain and codomain, and "$\mapsto$" only between the elements.



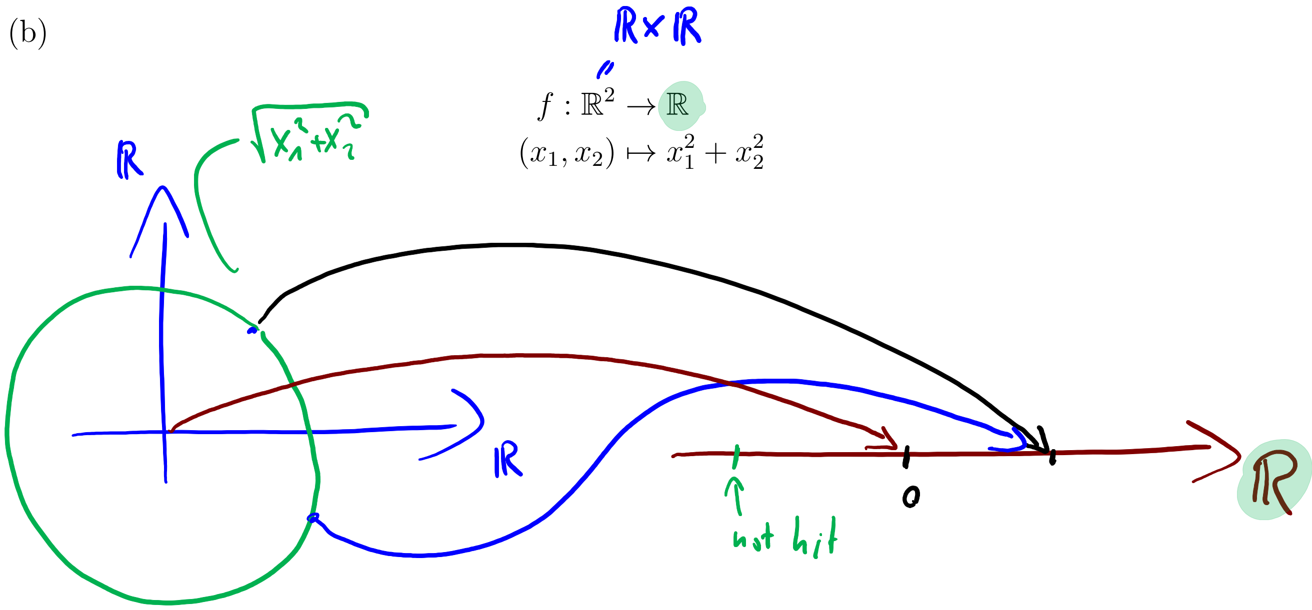
(Graph: $\{(x, f(x)) : x \in X\}$)

Example 1.32. (a) $f : \mathbb{N} \to \mathbb{N}$ with $f(x) = x^2$ maps each natural number to its square.



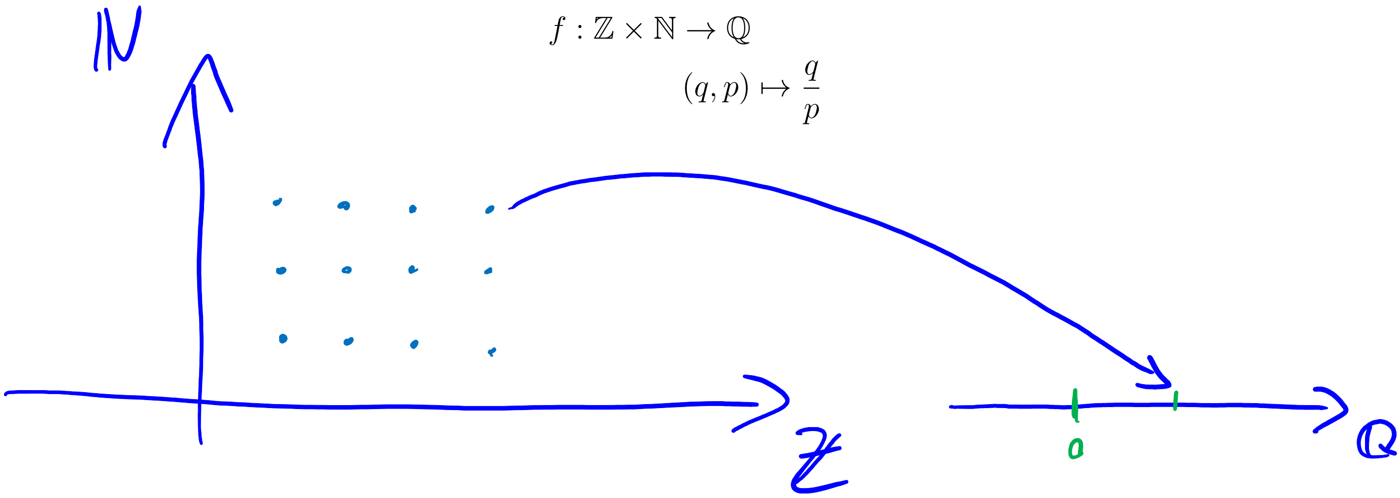
(b)

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \mapsto x_1^2 + x_2^2$$



(c)

$$f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$$
$$(q, p) \mapsto \frac{q}{p}$$



Well-definedness

What can go wrong with the definition of a map? Sometimes, when defining a function, it is not completely clear, if this makes sense. Then one has to work and make this function well-defined.

Example: the square-root

Try to define a map $a \to \sqrt{a}$ in a mathematically rigorous way.
Naive definition:

$$\begin{aligned} \sqrt{} : \mathbb{R} &\to \mathbb{R} \\ a &\mapsto \text{ the solution of } x^2 = a. \end{aligned}$$

Problem of well-definedness: As we all know, the above equation has two $(a > 0)$, one $(a = 0)$, or zero $(a < 0)$ solutions.

First way: restrict the domain of definition and the codomain

$$\mathbb{R}_0^+ = \{a \in \mathbb{R} : a \geq 0\} = [0, \infty)$$

Then:

$$\begin{aligned} \sqrt{} : \mathbb{R}_0^+ &\to \mathbb{R}_0^+ \\ a &\mapsto \text{ the non-negative solution of } x^2 = a. \end{aligned}$$

This yields the classical square-root.

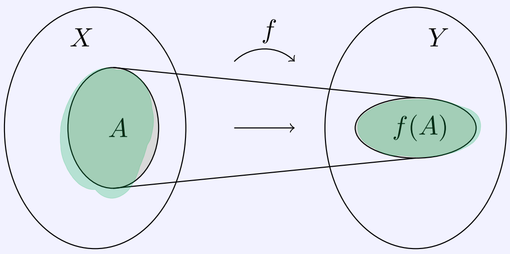
Image and preimage

For every well defined map $f : X \to Y$ and $A \subset X$, $B \subset Y$ we are interested in the following sets:

Definition 1.33.
Let $f : X \to Y$ be a function and $A \subset X$ and $B \subset Y$ some sets.

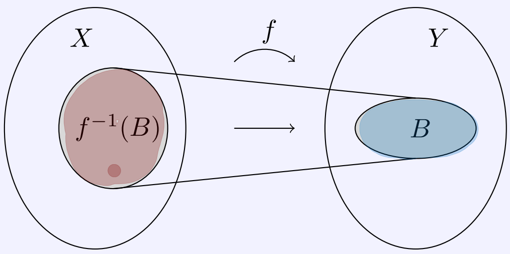
$$f(A) := \{f(x) : x \in A\}$$

is called the image of A under f.

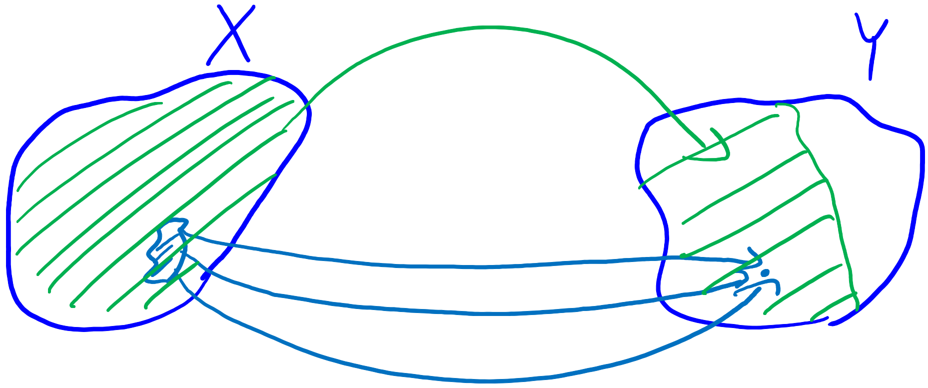


$$f^{-1}(B) := \{x \in X : f(x) \in B\}$$

is called the preimage of B under f.



not inverse function



Definition 1.34. Range and fiber

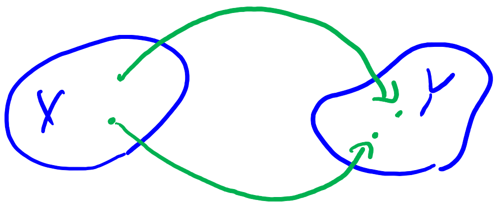
Let $f : X \to Y$ be a map.

$$f(X) := \mathrm{Ran}(f) := \{f(x) : x \in A\} \quad \textit{is called the range of } f.$$

For each $y \in Y$ the set:

$$f^{-1}(\{y\}) := \{x \in X : f(x) = y\} \quad \textit{is called a fiber of } f.$$

Injectivity, surjectivity, bijectivity, inverse

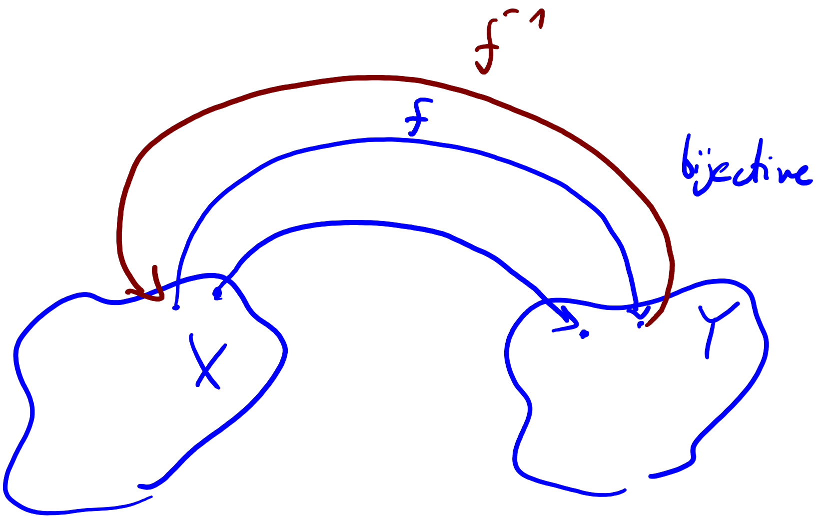


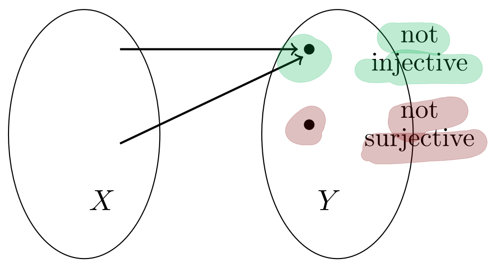
Definition 1.35. Injective, surjective and bijective

A map $f : X \to Y$ is called

- *injective if every fiber of f has only one element: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.*
- *surjective if $\mathrm{Ran}(f) = Y$. With quantifiers: $\forall y \in Y \; \exists x \in X : f(x) = y$.*
- *bijective if f is both injective and surjective.*

Example 1.36. Define the function that maps each student to her or his chair. This means that X is the set of all students in the room, and Y the set of all chairs in the room.





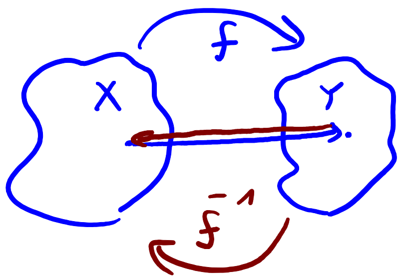
Rule of thumb: Surjective, injective, bijective

A map $f : X \to Y$ is

$$
\begin{aligned}
\textit{surjective} \;&\Leftrightarrow\; \textit{at each } y \in Y \textit{ arrives } \textbf{\textit{at least}} \textit{ one arrow}\\
&\Leftrightarrow\; f(X) = Y\\
&\Leftrightarrow\; \textit{the equation } f(x) = y \textit{ has for all } y \in Y \textit{ a solution}\\[1em]
\textit{injective} \;&\Leftrightarrow\; \textit{at each } y \in Y \textit{ arrives } \textbf{\textit{at most}} \textit{ one arrow}\\
&\Leftrightarrow\; (x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2))\\
&\Leftrightarrow\; (f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2)\\
&\Leftrightarrow\; \textit{the equation } f(x) = y \textit{ has for all } y \in f(X) \textit{ a } \textbf{\textit{unique}} \textit{ solution}\\[1em]
\textit{bijective} \;&\Leftrightarrow\; \textit{at each } y \in Y \textit{ arrives } \textbf{\textit{exactly}} \textit{ one arrow}\\
&\Leftrightarrow\; \textit{the equation } f(x) = y \textit{ has for all } y \in Y \textit{ a } \textbf{\textit{unique}} \textit{ solution}
\end{aligned}
$$

Thus, if f is bijective, there is a well defined inverse map

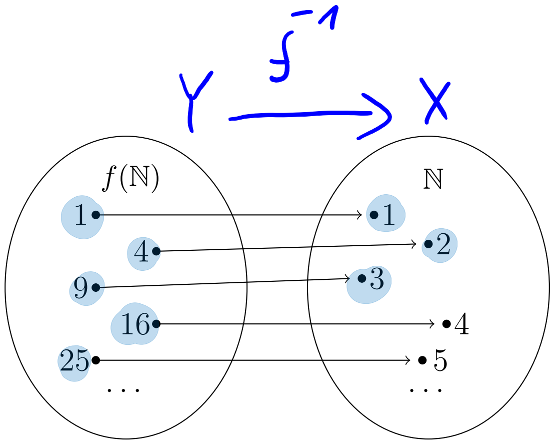
$$
\begin{aligned}
f^{-1} : Y &\to X\\
y &\mapsto x \text{ where } f(x) = y\,.
\end{aligned}
$$



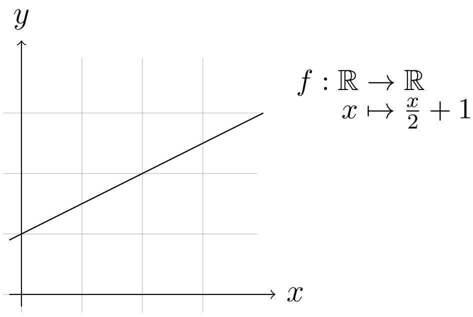
Then f is called *invertible* and f^{-1} is called the *inverse map* of f.

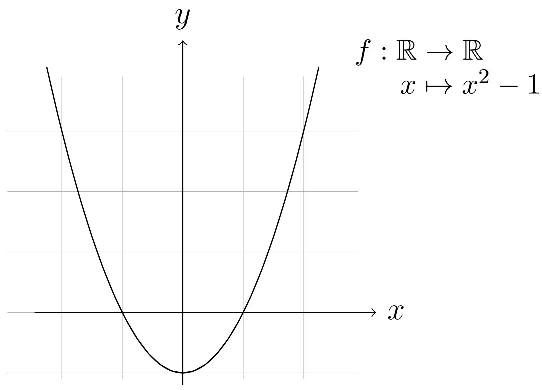
Example 1.37. Consider the function $f : \mathbb{N} \to \{1, 4, 9, 16, \dots\}$ given by $f(n) = n^2$. This is a bijective function. The inverse map f^{-1} is given by:

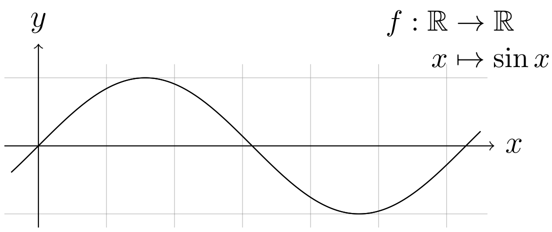
$$
\begin{aligned}
f^{-1} : \{1, 4, 9, 16, 25, \dots\} &\to \mathbb{N}\\
m &\mapsto \sqrt{m}\\
\text{or: } n^2 &\mapsto n
\end{aligned}
$$



Example 1.38. For a function $f : \mathbb{R} \to \mathbb{R}$, we can sketch the graph $\{(x, f(x)) : x \in X\}$ in the x-y-plane:







Which of the functions are injective, surjective or bijective?

Composition of maps

Definition 1.39.

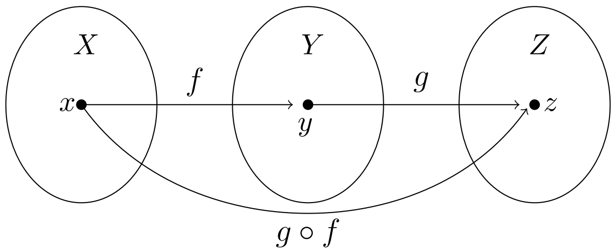
If $f : X \to Y$ and $g : Y \to Z$, we may compose, or concatenate these maps:

$$\begin{aligned} g \circ f : X &\to Z \\ x &\mapsto g(f(x)) \end{aligned}$$

We call $g \circ f$ the **composition** *of the two functions.*

Usually, $g \circ f \neq f \circ g$, the latter does not even make sense, in general.

$$X \to Y \to Z$$



Example 1.40. (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$; $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$

$$\begin{aligned} g \circ f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sin(x^2) \\ f \circ g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (\sin(x))^2 \end{aligned}$$

(b) Let X be a set. Then $id_X : X \to X$ with $x \mapsto x$ is called the *identity map*. If there is no confusion, one usually writes id instead of id_X. Let $f : X \to X$ be a function. Then

$$f \circ id = f = id \circ f.$$

1.4 Natural numbers and induction

The natural numbers are $\mathbb{N} = \{1, 2, 3 \ldots\}$.

- *Question 1:* When are two sets S, T of the same size? Have the same *cardinality* $|S| = |T|$? *Answer:* They have the same size if there is a bijective map $S \to T$.