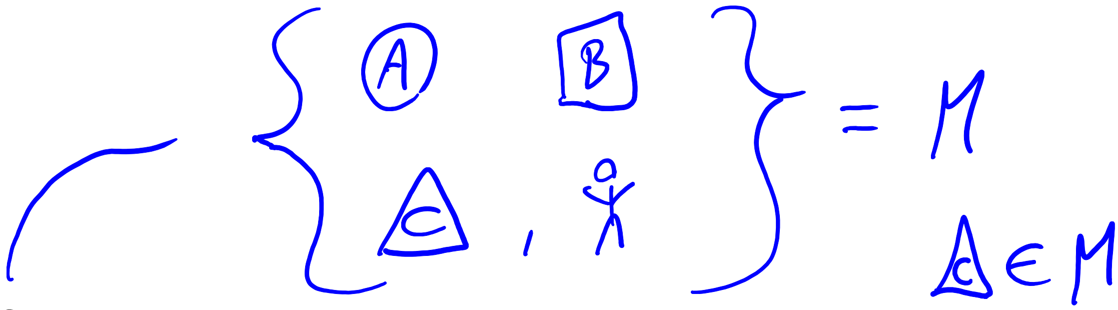


Sets

Definition 1.11. Set, element

A set is a collection into a whole of definite, distinct objects of our perception or of our thought. Such an object x of a set M is called an element of M and one writes $x \in M$. If x is not such an object of M, we write $x \notin M$.

A set is defined by giving all its elements

"$\neg(x \in M) = x \notin M$"

$M := \{1, 4, 9, 13\}$, $1 \in M$, $0 \notin M$

new symbol

The symbol ":=" is read as defined by *and means that the symbol M is newly introduced as a set by the given elements.*

Example 1.12.
- The empty set $\{\} = \emptyset = \varnothing$ is the unique set that has no elements at all.

  For all x, we have $x \notin \emptyset$
- The set that contains the empty set $\{\varnothing\}$, which is non-empty since it has exactly one element.

  $\emptyset \in \{\emptyset\}$
- A finite set of numbers is $\{1, 2, 3\} =: A$

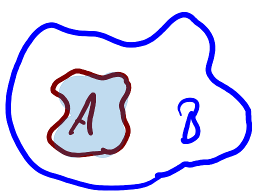
  three elements

  $A = \{1,3,2\} = \{3,2,1\} = \{1,1,1,2,3\}$

Notation 1.13. Let A, B be sets:
- $x \in A$ means x is an element of A
- $x \notin A$ means x is not an element of A
- $A \subset B$ means A is a subset of B: every element of A is contained in B
- $A \supset B$ means A is a superset of B: every element of B is contained in A

other symbol

$A \subseteq B$



$A \subset B$ means: $x \in A \Rightarrow x \in B$

- $A = B$ means $A \subset B \wedge A \supset B$. Note that the order of the elements does not matter in sets. If we want the order to matter, we rather define *tuples*: $(1, 2, 3) \neq (1, 3, 2)$. For sets, we always have $\{1, 2, 3\} = \{1, 3, 2\}$.
- $A \subsetneq B$ means A is a "proper" subset of B, every element of A is contained in B, but $A \neq B$.

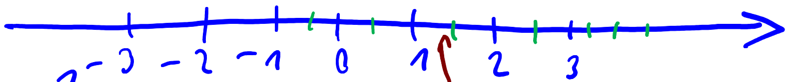
$\mathbb{Z}_+$, $\mathbb{Z}_-$

The important number sets

- $\mathbb{N}$ *is the set of the natural numbers* $1, 2, 3, \ldots$;
- $\mathbb{N}_0$ *is the set of the natural numbers and zero:* $0, 1, 2, 3, \ldots$;
- $\mathbb{Z}$ *is the set of the integers, which means* $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$;
- $\mathbb{Q}$ *is the set of the rational numbers, which means all fractions* $\frac{p}{q}$ *with* $p \in \mathbb{Z}$ *and* $q \in \mathbb{N}$;
- $\mathbb{R}$ *is the set of the real numbers (see next semester).*

$3 \cdot x = 1$

Other ways to define sets:



$A := \{n \in \mathbb{N} : 1 \leq n \leq 30\}$

$\sqrt{2} \notin \mathbb{Q}$

filling the line $\to \mathbb{R}$

$P(B) = \{A : A \subset B\}$ Power set

Definition 1.14. Cardinality

We use vertical bars $|\cdot|$ *around a set to denote the number of elements. For example, we have* $|\{1, 4, 9\}| = 3$. *The number of elements is called the* cardinality *of the set.*

Other notation $\# M = |M|$

What does ∞ mean?

Example 1.15. $|\{1, 3, 3, 1\}| = 2$, $|\{1, 2, 3, \ldots, n\}| = n$, $|\mathbb{N}| = \infty$ (?)

$|\mathbb{N}| < |\mathbb{R}|$

$|\mathbb{N}| = |\mathbb{Q}|$

Exercise 1.16. *Which of the following logical statements are true?*

$$3 \in \mathbb{N}, \quad 12034 \in \mathbb{N}, \quad -1 \in \mathbb{N}, \quad 0 \in \mathbb{N}, \quad 0 \in \mathbb{N}_0$$
$$-1 \in \mathbb{Z}, \quad 0 \notin \mathbb{Z}, \quad -2.7 \in \mathbb{Z}, \quad \tfrac{2}{3} \in \mathbb{Z},$$
$$\tfrac{2}{3} \in \mathbb{Q}, \quad -3 \in \mathbb{Q}, \quad -2.7 \in \mathbb{Q}, \quad \sqrt{2} \in \mathbb{Q},$$
$$\sqrt{2} \in \mathbb{R}, \quad \sqrt{-2} \in \mathbb{R}, \quad -\tfrac{2}{3} \in \mathbb{R}, \quad 0 \in \mathbb{R}.$$

$\to \sqrt{2}$ is irrational (proof by contradiction)

$x^2 = -2$

Predicates and quantifiers

Definition 1.17. Predicate

If X is any set and $A(x)$ is a logical statement depending on $x \in X$ (and true or false for every $x \in X$), we call $A(x)$ a predicate with variable x. Usually, one writes simply $A(x)$ instead of $A(x) = $ true.

$X = \mathbb{R}$, $A(x) = $ " $x < 0$ "

$M = \{x \in X : A(x)\} = \{x \in \mathbb{R} : x < 0\} = \{x \in \mathbb{R} \mid x < 0\}$

Definition 1.18. Quantifiers $\forall$ and $\exists$

We use $\forall$ ("for all") and $\exists$ ("it exists") and call them quantifiers. Moreover, we use the double point " : " inside the set brackets, which means "that fulfil".

The quantifiers and predicates are very useful for a compact notation:

- $\forall x \in X : A(x)$ *for all $x \in X$ $A(x)$ is true*
- $\exists x \in X : A(x)$ *there exists at least one $x \in X$ for which $A(x)$ is true*
- $\exists_1$ $\exists! x \in X : A(x)$ *there exists exactly one $x \in X$ for which $A(x)$ is true*

Negation of statements with quantifiers:

- $\neg(\forall x \in X : A(x)) \Leftrightarrow \exists x \in X : \neg A(x)$
- $\neg(\exists x \in X : A(x)) \Leftrightarrow \forall x \in X : \neg A(x)$

B = Every student has a seat in the lecture hall.

$\neg B$ = There is a student that has no seat in the lecture hall.

Example 1.19. There is no greatest natural number:

$$A(n) = \text{"}n \text{ is the greatest natural number"}$$

$$\neg(\exists n \in \mathbb{N} : A(n)) = \forall n \in \mathbb{N} : \neg A(n)$$

This true since $n+1 > n$ for all $n \in \mathbb{N}$.

$$\neg(\exists x \in X \ \forall y \in Y \ \exists z \in Z : A(x,y,z))$$

$$= \forall x \in X \ \exists y \in Y \ \forall z \in Z : \neg A(x,y,z)$$

Rule of thumb: Negation of the quantifier ($\forall$ and $\exists$)

"$\neg\forall = \exists\neg$" *and* "$\neg\exists = \forall\neg$"

Example 1.20. The set $M := \{x \in \mathbb{Z} : x^2 = 25\}$ is defined by the set of each integer x that squares to 25. We immediately see that this is just -5 and 5.

$$\{x \in \mathbb{Z} : x^2 = 25\} = \{-5, 5\}$$

$$\{x \in \mathbb{N} : x^2 = 25\} = \{5\}$$

$$\{x \in \mathbb{R} : x^2 = -25\} = \emptyset$$

You will see $|\{x \in \mathbb{C} : x^2 = -25\}| = 2$

Operations on sets

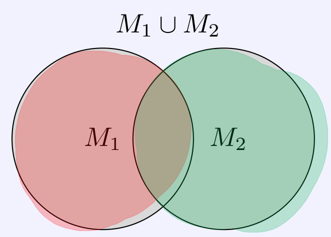
We remember the important operations for sets:

M_1, M_2 arbitrary sets

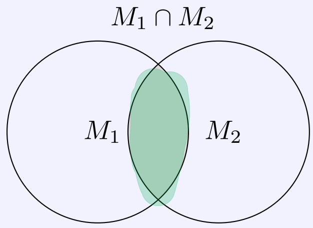
- $M_1 \cup M_2 := \{x : x \in M_1 \vee x \in M_2\}$ (union)
- $M_1 \cap M_2 := \{x : x \in M_1 \wedge x \in M_2\}$ (intersection)
- $M_1 \setminus M_2 := \{x : x \in M_1 \wedge x \notin M_2\}$ (set difference)

Definition 1.21. Set compositions

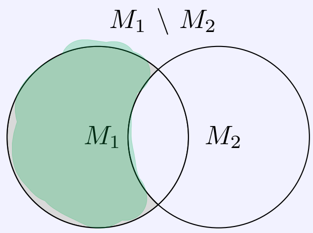
The union $M_1 \cup M_2$ is the new set that consists exactly of the objects that are elements of M_1 or M_2.



The intersection $M_1 \cap M_2$ *is the new set whose elements are the objects that are elements of* M_1 ***and*** M_2.

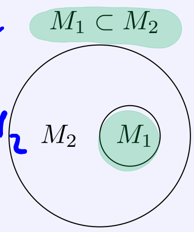


We write $M_1 \setminus M_2$ *for the set difference whose elements are the objects that are elements of* M_1 ***but not*** *elements of* M_2.



A subset of M_2 *is each set whose elements are also elements of* M_2.

$x \in M_1 \Rightarrow x \in M_2$

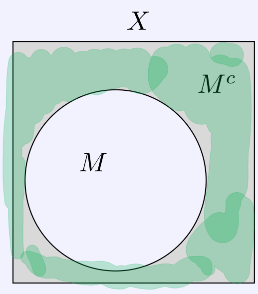


$\mathbb{N} \subset \mathbb{R}$

Definition 1.22. Complement set

Let X *be a set. Then for a subset* $M \subset X$ *there is a unique complement of* M *with respect to* X*:*

$$M^c := X \setminus M = \{x \in X : x \notin M\}$$



$X = \mathbb{N}$, $M = \{2, 4, 6, 8, 10, \dots\}$ even numbers $=: 2\mathbb{N}$

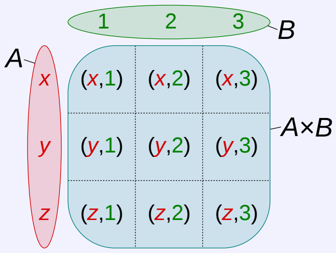
$M^c = \{1, 3, 5, 7, 9, 11, \dots\}$ odd numbers

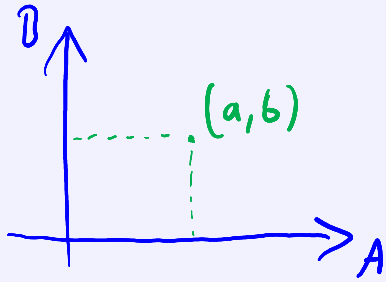
$M^c \cup M = X$

Definition 1.23. Product set

The Cartesian product *of two sets A, B is given as the set of all* pairs *(two elements with order):*

$$A \times B := \{(a, b) : a \in A, b \in B\} \quad \neq B \times A$$





(Source of the picture: Author Quartl - Wikipedia)

In the same sense, for sets $A_1, \dots, A_n$ the set of all n-tupels *is defined:*

$$A_1 \times \cdots \times A_n := \{(a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n\}$$

$\mathbb{N} \subset \mathbb{Z}$

Exercise 1.24. *Which statements are correct?*

$$\{1, 3\} \cup \{2, 4\} = \{1, 2, 4\}, \qquad \{1, 2\} \cup \{3, 4\} = \{3, 2, 4, 2, 1\}, \qquad \mathbb{N} \cup \mathbb{Z} = \mathbb{Z}.$$

$$\{1, 2, 4\} \cap \{3, 4, 5\} = \{4\}, \qquad \{1, 3\} \cap \{2, 4\} = \varnothing, \qquad \mathbb{N} \cap \mathbb{Z} = \mathbb{N}_0.$$

$$\{1, 2, 4\} \setminus \{3, 4, 5\} = \{1\}, \qquad \mathbb{N}_0 \setminus \mathbb{N} = \{0\}, \qquad \mathbb{N} \setminus \mathbb{Z} = \varnothing.$$

$$\mathbb{Z} \setminus \mathbb{N} = \{-x : x \in \mathbb{N}\}, \qquad \mathbb{N} \subset \mathbb{N}_0, \qquad \mathbb{Z} \subset \mathbb{N}_0, \qquad (\mathbb{Z} \setminus \mathbb{Q}) \subset \mathbb{N}.$$

zero is missing

$$\mathbb{N} \subset \mathbb{N}, \qquad -3 \in \mathbb{Z} \setminus \mathbb{N}_0, \qquad \tfrac{3}{7} \in \mathbb{Q} \setminus \mathbb{Z}, \qquad \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q},$$

Exercise 1.25. *Which claims are correct? Prove or give a counterexample. Here, we solve one exercise in detail!*

(a) $(\mathbb{Q} \setminus \mathbb{R}) \subset \mathbb{N}_0$.

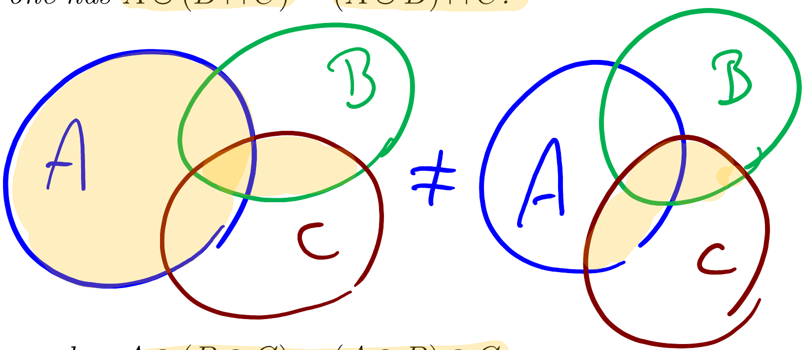
$\emptyset$

$\emptyset \subset A$ for every set A

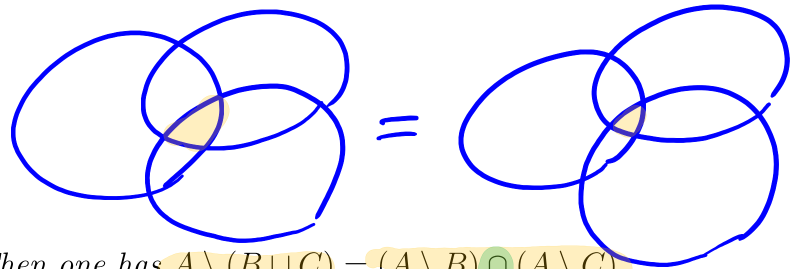
$x \in \emptyset \Rightarrow x \in A$ ✓

false

(b) *Let* A, B, C *be three sets. Then one has* $A \cup (B \cap C) = (A \cup B) \cap C$.



(c) *Let* A, B, C *be three sets. Then one has* $A \cap (B \cap C) = (A \cap B) \cap C$.



(d) *Let* A, B, C *be three sets. Then one has* $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

To show that $X = Y$, you should show $X \subset Y$ and $X \supset Y$.

$$x \in A \setminus (B \cup C) \Leftrightarrow x \in A \wedge x \notin (B \cup C)$$
$$\Leftrightarrow x \in A \wedge x \notin B \wedge x \notin C$$
$$\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$$
$$\Leftrightarrow x \in A \setminus B \cap A \setminus C \qquad \Rightarrow A \setminus (B \cup C) = A \setminus B \cap A \setminus C$$

□

Exercise 1.26. *Describe the following sets and calculate its cardinalities:*

(a) $X_1 := \{x \in \mathbb{N} \colon \exists a, b \in \{1, 2, 3\}$ *with* $x = a - b\}$

$X_1 = \{ \dots \}$

(b) $X_2 := \{(a - b) \colon a, b \in \{1, 2, 3\}\}$

(c) $X_3 := \{|a-b|: a, b \in \{1,2,3\}\}$

(d) $X_4 := \{1, ..., 20\} \setminus \{n \in \mathbb{N}: \exists a, b \in \mathbb{N} \text{ with } 2 \leq a \text{ and } 2 \leq b \text{ and } n = a \cdot b\}$.

(e) $X_5 := \{S: S \subset \{1,2,3\}\}$. , Subsets of $\underbrace{\{1,2,3\}}_{\text{size}=3}$?

$= \{\emptyset, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$|X_5| = 8 = 2^3$

1.2 Real Numbers

t.me/AnonymeInformatiker

Some *laws* apply:

$$a + (b + c) = (a + b) + c, \qquad a(bc) = (ab)c \qquad \text{associative law}$$

$$a + b = b + a \qquad ab = ba \qquad \text{commutative law}$$