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Foundations of mathematics

1.1 Logic and sets

Definition 1.1. logical statement, proposition

A logical statement (or proposition) is a statement, which means a meaningful declarative sentence, that is either true or false.

- no opinions, - no questions, no alternative fact

- no self-contradictory statements

Example 1.2. Which of these are logical statements?

(a) Hamburg is a city. Yes, it is true.

(b) $1 + 1 = 2$. Yes, it is true

(c) The number 5 is smaller than the number 2. Yes, it is false

(d) Good morning! No!

(e) $x + 1 = 1$. No! It will get one, if x gets a value. (predicate)

(f) Today is Tuesday. No! (predicate)

Logical operations

In the following, we will consider two logical statements A and B.

Definition 1.3. Negation $\neg A$ ("not A")

$\neg A$ is true if and only if A is false.

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Truth table

A	$\neg A$
T	F
F	T

(1.1)

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Example 1.4. What are the negations of the following logical statements?

(a) The wine bottle is full. $= A$

$\neg A =$ The wine bottle is not full.

$\rightarrow$ It is not the same as: The wine bottle is empty.

(b) The number 5 is smaller than the number 2. $= A = 5 < 2$

$\neg A =$ 5 is greater or equal to 2.

$\neg A = 5 \geq 2$

(c) All students are in the lecture hall. $= A$

$\neg A =$ Not all students are in the lecture hall.

$=$ There is a student that is not in the lecture hall.

Definition 1.5. Conjunction $A \wedge B$ ("A and B")

$A \wedge B$ is true if and only if both A and B are true.

Truth table

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

(1.2)

Definition 1.6. Disjunction $A \vee B$ ("A or B")

$A \vee B$ is true if and only if at least one of A or B is true.

Truth table (1.3)

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.7. Conditional $A \to B$ ("If A then B")

$A \to B$ is only false if A is true but B is false.

Truth table (1.4)

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

Definition 1.8. Biconditional $A \leftrightarrow B$ ("A if and only if B")

$A \leftrightarrow B$ is true if and only if $A \to B$ and $B \to A$ is true.

Truth table (1.5)

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

If a conditional or biconditional is true, we have a short notation for this that is used throughout the whole field of mathematics:

Definition 1.9. Implication and equivalence

If $A \to B$ is true, we call this an implication and write:

$$A \Rightarrow B\,.$$

If $A \leftrightarrow B$ is true, we call this an equivalence and write:

$$A \Leftrightarrow B\,.$$

This means that we speak of *equivalence* of A and B if the truth values in the truth table are exactly the same. For example, we have

$$A \leftrightarrow B \quad \Leftrightarrow \quad (A \to B) \wedge (B \to A) .$$

Now one can ask: *What to do with truth-tables?*

Truth table (1.6)

A	B	$\neg A$	$\neg B$	$\neg B \to \neg A$	$A \to B$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Therefore:

$$(A \to B) \quad \Leftrightarrow \quad (\neg B \to \neg A) .$$

This is the *proof by contraposition*:

Contraposition

If $A \Rightarrow B$, then also $\neg B \Rightarrow \neg A$.

Rule of thumb: Contraposition

To get the contraposition $A \Rightarrow B$, you should exchange A and B and set a $\neg$-sign in front of both: $\neg B \Rightarrow \neg A$.
It is clear: The contraposition of the contraposition is again $A \Rightarrow B$.

If there is fog, we have poor visibility.
$A \to B$

If there is no poor visibility, then there is no fog.
$\neg B \to \neg A$

The contraposition is an example of a deduction rule, which basically tells us how to get new true proposition from other true propositions. The most important deduction rules are given just by using the implication.

Modus ponens

If $A \Rightarrow B$ and A is true, then also B is true.

A	B	$A \to B$	$A \wedge (A \to B) \to B$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

tautology

Chain syllogism

If $A \Rightarrow B$ and $B \Rightarrow C$, then also $A \Rightarrow C$.

A	B	C	
T			
T			
T			
T			
F			
F			

Reductio ad absurdum

If $A \Rightarrow B$ and $A \Rightarrow \neg B$, then $\neg A$ is true.

A	B	$A \to B$	$A \to \neg B$	$\neg A$
T	T	T	F	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

Exercise 1.10. *Let* "All birds can fly" *be a true proposition (axiom). Are the following deductions correct?*

$Bi \Rightarrow F$

- *If Seagulls are birds, then Seagulls can fly.*

  $S \Rightarrow Bi$ $\qquad$ $S \Rightarrow F$

- *If Penguins are birds, then Penguins can fly.*

  $P \Rightarrow Bi$ $\qquad$ $P \Rightarrow F$

- *If Butterflies are birds, then Butterflies can fly.*

  $Bu \Rightarrow Bi$ $\qquad$ $Bu \Rightarrow F$

- *If Butterflies can fly, then Butterflies are birds.*

  $Bu \Rightarrow F$ $\qquad$ $Bu \Rightarrow Bi$