SoSe 2019
Hamburg, 16.09.2019

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Final Exam for Linear Algebra

Problem 1: **(1 Point)**

a) State your surename, forename, Matrikelnummer, subject and semester at the front cover. Also give your signature. Check if all 8 problems actually are in your exam.

b) What is the dimension of the kernel for the following matrix $A \in \mathbb{R}^{4\times 4}$?

$$A = \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim(\mathrm{Ker}(A)) =$	0	1	2	3	4	5	6	7	8
	☐	☐	☐	☐	☐	☐	☐	☐	☐

Solution. That is clear! ☐

Problem 2: **(5 Points)**

The following four vectors in $\mathbb{R}^3$ are given.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

a) Calculate the three-dimensional volume $\mathrm{Vol}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

b) Is the family $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ linearly dependent? Justify your answer.

c) Determine a basis and the dimension of $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

Solution. (a)

$$\mathrm{Vol}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \det \begin{pmatrix} 0 & 2 & 4 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix} =: V$$

(1 point)

For example, expanding along the first column:

$$V = -2 \det \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}$$

And then use the determinant formula for 2×2-matrices.

$$V = -8 + 8 = 0\,.$$

(1 point)

(b) The family is linearly dependent because $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ span a volume of zero. (Alternatively: Four vectors in $\mathbb{R}^3$ are always linearly dependent.) (1 point)

(c) We can calculate $\mathrm{Vol}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$ analogously to before and get

$$\mathrm{Vol}_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = -2 \det \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = (-2) \cdot 12 + 0 \neq 0\,.$$

This tells us that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$ are linearly independent. Combining this with (b), we get that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$ form a basis of $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ and the dimension is 3. (2 points) □

Problem 3: **(6 Points)**

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

a) Determine the characteristic polynomial of A and calculate the eigenvalues of A. State also the algebraic multiplicities.

b) For each eigenvalue, determine a basis for the corresponding eigenspace. State also the geometric multiplicities.

c) Why is A diagonalisable? Find an invertible matrix $X \in \mathbb{R}^{3\times 3}$ and a diagonal matrix $D \in \mathbb{R}^{3\times 3}$ such that $X^{-1}AX = D$.

d) Calculate $A^2 X$.

Solution. (a) Calculate:

$$\det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & -1-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda)(2-\lambda)$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$, where the algebraic multiplicity is one for all three. (1 point)
(b)

$$\operatorname{Ker}(A - \lambda_1 \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 3 \end{pmatrix} \overset{\binom{II:(-2)}{III-3\cdot II}}{=} \operatorname{Ker} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$$

$$\operatorname{Ker}(A - \lambda_2 \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 0 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \overset{\binom{II+I}{III+II'}}{=} \operatorname{Ker} \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\operatorname{Ker}(A - \lambda_3 \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} -1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \right)$$

The basis is always given in the span and the geometric multiplicity is one for all eigenvalues. (2 points)

(c) Because the geometric multiplicities coincide with algebraic multiplicities, A is diagonalisable. (1 point).

The wanted matrices in $X^{-1}AX = D$ are:

$$X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(1 point)

(d) Because $A^2X = A(AX) = A(XD) = (AX)D = XD^2$, we just have to calculate D^2:

$$D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow XD^2 = \begin{pmatrix} 1 & 1 & 4 \\ -1 & 0 & 8 \\ 0 & 0 & -12 \end{pmatrix}$$

(1 point) □

Problem 4: **(9 Points)**

Decide if the following claims are true or false and mark it with a cross. (Marking both true and false will not get you a point. You do **not** get minus points for wrong answers.)

	false	true
If $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, then A^T is not an invertible matrix.	X	
If $\mathbf{v}$ is an eigenvector for the matrix A, then it is also an eigenvector for $A + 2A$.		X
For all matrices $A \in \mathbb{R}^{m\times n}$, one has $\operatorname{Ran}(A^T A) = \mathbb{R}^n$.	X	
A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\operatorname{Ker}(A) = \{\mathbf{o}\}$.		X
The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ is invertible.		X
All matrices $A \in \mathbb{R}^{m\times k}$ satisfy $\dim(\operatorname{Ran}(A)) + \dim(\operatorname{Ker}(A)) = k$.		X
Each matrix $A \in \mathbb{C}^{n\times n}$ has at least one eigenvalue.		X
For an inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^3$, the set $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x}, \mathbf{o}\rangle = 0\}$ is a linear subspace.		X
Each matrix $A \in \mathbb{C}^{n\times n}$ is similar to a diagonal matrix.	X	

Solution. See above. □

Problem 5: **(5 Points)**

Let V be an $\mathbb{F}$-vector space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Moreover, let $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ be an orthonormal basis of V.

a) Let $\mathbf{x}, \mathbf{y} \in V$ be given as $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_4 - 2\mathbf{b}_8 - 2\mathbf{b}_9 + \mathbf{b}_{10}$ and $\mathbf{y} = 2\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_5 - 3\mathbf{b}_7 - \mathbf{b}_9$. Calculate $\langle \mathbf{x}, \mathbf{y} \rangle$.

b) Calculate the orthogonal projection of $\mathbf{x}$ onto $\mathrm{Span}(\mathbf{y})$.

Solution. (a) In general for all $\mathbf{x}$ and $\mathbf{y}$ with

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$$

$$\mathbf{y} = \beta_1 \mathbf{b}_1 + \cdots + \beta_n \mathbf{b}_n \,,$$

we get

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n), \mathbf{y} \rangle = \sum_{j=1}^{n} \alpha_j \langle \mathbf{b}_j, \mathbf{y} \rangle = \sum_{j=1}^{n} \alpha_j \overline{\beta_j} \,.$$

Hence $\langle \mathbf{x}, \mathbf{y} \rangle = 3 \cdot 2 + (-2)(-1) = 8$. (2 points)

(b) First step is normalisation of $\mathbf{y}$:

$$\hat{\mathbf{y}} := \frac{1}{\|\mathbf{y}\|} \mathbf{y} = \frac{1}{\sqrt{16}} \mathbf{y}$$

(1 point)

Then, we calculate

$$\langle \mathbf{x}, \hat{\mathbf{y}} \rangle \hat{\mathbf{y}} = \langle \mathbf{x}, \mathbf{y} \rangle \frac{1}{\|\mathbf{y}\|^2} \mathbf{y} = \frac{8}{16} \mathbf{y} = \frac{1}{2} \mathbf{y}.$$

(1 point)

And then:

$$\mathbf{x}|_{\mathrm{Span}(\mathbf{y})} = \frac{1}{2} \mathbf{y} = \mathbf{b}_1 + \frac{1}{2} \mathbf{b}_2 + \frac{1}{2} \mathbf{b}_5 - \frac{3}{2} \mathbf{b}_7 - \frac{1}{2} \mathbf{b}_9$$

(1 point) □

Problem 6: **(5 Points)**

Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of real polynomials with degree at most 2. For $k \in \mathbb{N}_0$, we denote the monomials by $\mathbf{m}_k : \mathbb{R} \to \mathbb{R}$, which means $\mathbf{m}_k(x) := x^k$ for all $x \in \mathbb{R}$. Let $\ell : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$ be a linear map given by

$$\ell\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) := (x_1 + x_2 + x_3)\mathbf{m}_1 + (x_1 - x_2 - 2x_3)\mathbf{m}_0 .$$

a) Determine the matrix representation of ℓ with respect to the following bases

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right), \qquad \mathcal{C} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2) .$$

b) We introduce two new bases

$$\mathcal{B}' = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right), \qquad \mathcal{C}' = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2 - \mathbf{m}_1) .$$

Calculate the change of basis matrices $T_{\mathcal{B} \leftarrow \mathcal{B}'}$ and $T_{\mathcal{C} \leftarrow \mathcal{C}'}$.

c) Calculate the matrix representation $\ell_{\mathcal{C}' \leftarrow \mathcal{B}'}$ by using part a) and b).

Solution. (a) We first calculate:

$$\begin{aligned} \ell(\mathbf{b}_1) &= 1\mathbf{m}_1 + 1\mathbf{m}_0 = 1\mathbf{m}_0 + 1\mathbf{m}_1 + 0\mathbf{m}_3 \\ \ell(\mathbf{b}_2) &= 1\mathbf{m}_1 + (-1)\mathbf{m}_0 = -1\mathbf{m}_0 + 1\mathbf{m}_1 + 0\mathbf{m}_3 \\ \ell(\mathbf{b}_3) &= 1\mathbf{m}_1 + (-2)\mathbf{m}_0 = -2\mathbf{m}_0 + 1\mathbf{m}_1 + 0\mathbf{m}_3 \end{aligned}$$

The matrix representation $\ell_{\mathcal{C} \leftarrow \mathcal{B}}$ is then:

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

(2 points)

(b) The change of basis matrices are immediately given:

$$T_{\mathcal{B} \leftarrow \mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_{\mathcal{C} \leftarrow \mathcal{C}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

(1 point)

(c) Using the formula

$$\ell_{\mathcal{C}' \leftarrow \mathcal{B}'} = T_{\mathcal{C}' \leftarrow \mathcal{C}} \ell_{\mathcal{C} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \mathcal{B}'}$$

(1 point)

Since $T_{\mathcal{C}' \leftarrow \mathcal{C}} = (T_{\mathcal{C} \leftarrow \mathcal{C}'})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we get:

$$\ell_{\mathcal{C}' \leftarrow \mathcal{B}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -1 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(1 point) □

Problem 7: **(4 Points)**

Let

$$A := \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 2} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \\ \frac{5}{2} \end{pmatrix} \in \mathbb{R}^4 .$$

a) Calculate $A^T A$ and $A^T \mathbf{b}$.

b) Find the only vector $\mathbf{x} \in \mathbb{R}^2$ that solves $A^T A\mathbf{x} = A^T \mathbf{b}$.

c) Calculate $\|A\mathbf{x} - \mathbf{b}\|$ for the vector $\mathbf{x} \in \mathbb{R}^2$ from b).

Hint: $\|\cdot\|$ *denotes the standard euclidean norm in* $\mathbb{R}^4$.

d) Show that $A\mathbf{x} = \mathbf{b}$ has no solution. In order to show this, you can use b) and c).

Solution. (a) Calculate $A^T A = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$ and $A^T \mathbf{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$. (1 point)

(b) Solve the system $\left(\begin{array}{cc|c} 6 & 2 & 5 \\ 2 & 4 & 0 \end{array}\right)$ to $\left(\begin{array}{cc|c} 6 & 2 & 5 \\ 0 & 10 & -5 \end{array}\right)$ to get $\mathbf{x} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$ as the unique solution. (1 point)

(c) We calculate

$$A\mathbf{x} - \mathbf{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

Therefore the norm is given by $\|A\mathbf{x} - \mathbf{b}\| = \sqrt{9+1+1+9} = 2\sqrt{5}$ (1 point)

(d) All solutions of $A\mathbf{x} = \mathbf{b}$ have to be solution of $A^T A\mathbf{x} = A^T\mathbf{b}$ as well. Hence, we only check if $\mathbf{x}$ from (b) is a solution. However (c) tells us it is not. (1 point) □

Problem 8: **(5 Points)**

Let $A \in \mathbb{C}^{n\times n}$ be selfadjoint with the property $\operatorname{spec}(A) \subset (0, \infty)$.

a) Show that $\langle A\mathbf{x}, \mathbf{x}\rangle_{\text{euclid}} > 0$ for all $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{o}\}$. How does one call such matrices with this property?

Hint: $\langle\cdot,\cdot\rangle_{\text{euclid}}$ *denotes the standard inner product in* $\mathbb{C}^n$.

b) Is A invertible? Justify your answer.

c) Give an example of such a matrix $A \in \mathbb{C}^{3\times 3}$.

Solution. (a) Since A is selfadjoint, it is diagonalisable with diagonal matrix D and unitary X such that $X^*AX = D$. In the D we find the eigenvalues of A, denoted by $\lambda_1, \dots, \lambda_n$, and we can calculate:

$$\langle A\mathbf{x}, \mathbf{x}\rangle_{\text{euclid}} = \langle XDX^*\mathbf{x}, \mathbf{x}\rangle_{\text{euclid}} = \langle D\mathbf{x}, \mathbf{x}\rangle_{\text{euclid}}$$

because X is unitary. Then we get for $\mathbf{x} = (x_1, \dots, x_n)^T$ immediately:

$$\langle D\mathbf{x}, \mathbf{x}\rangle_{\text{euclid}} = \sum_{j=1}^{n} \lambda_j |x_j|^2 \geq 0$$

It is only 0 if all x_j are zero, which is the zero vector $\mathbf{o}$. (2 points)

Such matrices are called positive definite. (1 point)

(b) A is invertible because all eigenvalues are positive. (1 point)

(c) One example is $A = \mathbb{1}_3$. (1 point) □