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Final Exam for Linear Algebra

Problem 1: **(1 Point)**

a) State your surename, forename, Matrikelnummer, subject and semester at the front cover. Also give your signature. Check if all 8 problems actually are in your exam.

b) What is the rank of the following matrix $A \in \mathbb{R}^{3\times 3}$?

$$A = \begin{pmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{pmatrix}$$

rank(A) =	0	1	2	3	4	5	6	7	8
	☐	☐	☐	☐	☐	☐	☐	☐	☐

Solution. That is clear! ☐

Problem 2: **(5 Points)**

The following four vectors span a four-dimensional parallelepiped in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix}.$$

a) Calculate the four-dimensional volume $\mathrm{Vol}_4(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

b) Is the family $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ linearly dependent? Justify your answer.

c) Is the matrix

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

invertible? Justify your answer.

Solution. (a)

$$\mathrm{Vol}_4(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \det \begin{pmatrix} 2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -2 \end{pmatrix} =: V$$

(1 point)

For example, expanding along the first column:

$$V = 2 \det \begin{pmatrix} -2 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & -3 & -2 \end{pmatrix} - (-1) \det \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & -3 & -2 \end{pmatrix}$$

And then use Sarrus:

$$V = 2(-5) + 8 = -2\,.$$

(2 points)

(b) The family is not linearly dependent because the volume they span is not zero. (1 point)

(c) The matrix is invertible because the determinant is $2 \neq 0$. Just the last two columns are switched in the determinant from A (in contrast to (a)). (1 point) $\square$

Problem 3: **(6 Points)**

Let

$$A = \begin{pmatrix} 5 & -4 & 2 \\ 4 & -3 & 2 \\ -2 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

a) Show that $\lambda = 1$ is an eigenvalue of A with geometric multiplicity 2, and determine a basis for the corresponding eigenspace.

b) Show that $\mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of A, and calculate the associated eigenvalue.

c) Show that A is diagonalisable and find an invertible matrix $X \in \mathbb{R}^{3\times 3}$ and a diagonal matrix $D \in \mathbb{R}^{3\times 3}$ such that $X^{-1}AX = D$.

d) Find a 3×3-matrix B such that $B^2 = A$.

Solution. (a) Calculate:

$$\operatorname{Ker}(A - 1\mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 4 & -4 & 2 \\ 4 & -4 & 2 \\ -2 & 2 & -1 \end{pmatrix} \overset{\binom{II-I}{III+I/2}}{=} \operatorname{Ker} \begin{pmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span} \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

Kernel is not trivial, hence $\lambda_1 = 1$ is an eigenvalue and the two vectors inside the span define a basis of the corresponding eigenspace. (2 points)

(b) Matrix-vector multiplication:

$$A\mathbf{v} = \begin{pmatrix} 5 & -4 & 2 \\ 4 & -3 & 2 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore $\mathbf{v} \in \operatorname{Ker}(A)$ which means $\mathbf{v}$ is an eigenvector with corresponding eigenvalue $\lambda_2 := 0$. (1 point)

(c) By part (a) and by part (b), we find a basis consisting of eigenvectors: $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \right)$ (Or: $\gamma(\lambda_1) + \gamma(\lambda_2) = 3 = n$).

The wanted matrices in $X^{-1}AX = D$ are:

$$X = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(2 points)}$$

(d) Because $D^2 = D$ and $A^2 = XDX^{-1}XDX^{-1} = XD^2X^{-1} = XDX^{-1} = A$, we can choose $B = A$. (1 point) $\square$

Problem 4: **(9 Points)**

Decide if the following claims are true or false and mark it with a cross. (Marking both true and false will not get you a point. You do **not** get minus points for wrong answers.)

	false	true
Each matrix $A \in \mathbb{C}^{n\times n}$ is similar to a triangular matrix.		X
For a matrix $A \in \mathbb{C}^{n\times n}$, the following holds: If $\mathrm{spec}(A)$ lies on the unit circle in $\mathbb{C}$, then A is not invertible.	X	
A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \neq 0$.		X
If $\mathbf{v}$ is an eigenvector for the matrix A, then it is also an eigenvector for A^3.		X
Each $A \in \mathbb{R}^{3\times 3}$ has at least one real eigenvalue.		X
If $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then A^T is also an orthogonal matrix.		X
There is a matrix $A \in \mathbb{R}^{4\times 5}$ that satisfies $\dim(\mathrm{Ran}(A)) = 2$ and $\dim(\mathrm{Ker}(A)) = 3$.		X
Each matrix $A \in \mathbb{C}^{n\times n}$ has infinitely many eigenvectors.		X
The matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ is diagonalisable.	X	

Solution. See above. □

Problem 5: **(5 Points)**

Let V be an $\mathbb{F}$-vector space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Moreover, let $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ be an orthonormal basis of V.

a) Show that for all $\mathbf{x}, \mathbf{y} \in V$ the following holds:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{b}_j \rangle \langle \mathbf{b}_j, \mathbf{y} \rangle \,.$$

b) Let $\mathbf{x} \in V$ be given as $\mathbf{x} = 2\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_5 - 3\mathbf{b}_7 - \mathbf{b}_9$. Calculate the norm $\|\mathbf{x}\|$.

Solution. (a) Since $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ is a basis, we can write $\mathbf{x}$ in a unique way as linear combination:

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n \,.$$

We also know that $\alpha_j = \langle \mathbf{x}, \mathbf{b}_j \rangle$ because we have an ONB. (1 point)

Then we calculate:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (\alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n), \mathbf{y} \rangle = \sum_{j=1}^{n} \alpha_j \langle \mathbf{b}_j, \mathbf{y} \rangle = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{b}_j \rangle \langle \mathbf{b}_j, \mathbf{y} \rangle \,.$$

(2 points)

(b) Here, we can use the formula from (a):

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{b}_j \rangle \langle \mathbf{b}_j, \mathbf{x} \rangle = 4 + 1 + 1 + 9 + 1 = 16$$

Hence, we get $\|\mathbf{x}\| = 4$. (2 points) $\square$

Problem 6: **(5 Points)**

Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of real polynomials with degree at most 2. For $k \in \mathbb{N}_0$, we denote the monomials by $\mathbf{m}_k : \mathbb{R} \to \mathbb{R}$, which means $\mathbf{m}_k(x) := x^k$ for all $x \in \mathbb{R}$. Let $\ell : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$ and $k : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2$ be linear maps given by

$$\ell\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) := (x_1 + x_2 + x_3)\mathbf{m}_0 + (x_1 + x_2)\mathbf{m}_1 + x_1\mathbf{m}_2$$

and

$$k(\mathbf{p}) := \begin{pmatrix} \mathbf{p}'(1) \\ \mathbf{p}(1) - \mathbf{p}''(1) \end{pmatrix}.$$

Determine the matrix representation of $k \circ \ell$ with respect to the following basis $\mathcal{B}$ in $\mathbb{R}^3$ and the basis $\mathcal{C}$ in $\mathbb{R}^2$:

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right), \qquad \mathcal{C} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right).$$

Solution. Helpful to calculate first $k(\mathbf{m}_0) = \binom{0}{1}$, $k(\mathbf{m}_1) = \binom{1}{1}$ and $k(\mathbf{m}_2) = \binom{2}{-1}$. Calculate the images of the basis vectors under $k \circ \ell$:

$$(k \circ \ell)(\mathbf{b}_1) = k(2\mathbf{m}_0 + 2\mathbf{m}_1 + 1\mathbf{m}_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$(k \circ \ell)(\mathbf{b}_2) = k(1\mathbf{m}_0 + 1\mathbf{m}_1 + 0\mathbf{m}_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$(k \circ \ell)(\mathbf{b}_3) = k(2\mathbf{m}_0 + 1\mathbf{m}_1 + 0\mathbf{m}_2) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \quad \text{(3 points)}$$

Then we get for the canonical basis $\mathcal{E}$ in $\mathbb{R}^2$:

$$(k \circ \ell)_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 4 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix}.$$

Since $T_{\mathcal{C} \leftarrow \mathcal{E}} = (T_{\mathcal{E} \leftarrow \mathcal{C}})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we get:

$$(k \circ \ell)_{\mathcal{C} \leftarrow \mathcal{B}} = T_{\mathcal{C} \leftarrow \mathcal{E}}(k \circ \ell)_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 4 & 1 & 1 \\ -3 & -2 & -3 \end{pmatrix}. \quad \text{(2 points)}$$

Alternatively, calculate the matrix representations separately:

Let $\mathcal{E}_2$ be the canonical basis in $\mathbb{R}^2$, $\mathcal{E}_3$ be the canonical basis in $\mathbb{R}^3$ and $\mathcal{M} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ be the monomial basis in $\mathcal{P}_2(\mathbb{R})$. Then, we get:

$$(k)_{\mathcal{E}_2 \leftarrow \mathcal{M}} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix},$$

$$(\ell)_{\mathcal{M} \leftarrow \mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad \text{(2 points)}$$

Using the formula

$$(k \circ \ell)_{\mathcal{C} \leftarrow \mathcal{B}} = T_{\mathcal{C} \leftarrow \mathcal{E}_2} (k)_{\mathcal{E}_2 \leftarrow \mathcal{M}} (\ell)_{\mathcal{M} \leftarrow \mathcal{E}_3} T_{\mathcal{E}_3 \leftarrow \mathcal{B}}, \quad \text{(1 point)}$$

we get:

$$(k \circ \ell)_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 4 & 1 & 1 \\ -3 & -2 & -3 \end{pmatrix}. \quad \text{(2 points)}$$

$\square$

Problem 7: **(5 Points)**

Let

$$A := \begin{pmatrix} -1 & -2 \\ -2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathbb{R}^{3\times 2} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \in \mathbb{R}^3 .$$

a) Find all $\mathbf{x} \in \mathbb{R}^2$ that solve $A^T A\mathbf{x} = A^T\mathbf{b}$.

b) Show that $A\mathbf{x} = \mathbf{b}$ has no solution. In order to show this, you can use (a).

c) Calculate the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Ran}(A)$ (with respect to the standard inner product in $\mathbb{R}^3$).

Solution. (a) First, calculate $A^T A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$ and $A^T\mathbf{b} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$. (1 point)

Then solve the system to get $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as the unique solution. (1 point)

(b) All solutions of $A\mathbf{x} = \mathbf{b}$ have to be solution of $A^T A\mathbf{x} = A^T\mathbf{b}$ as well. Hence, we only check if $\mathbf{x}$ from (a) is a solution:

$$A\mathbf{x} = \begin{pmatrix} ? \\ ? \\ -1 \end{pmatrix} \neq \mathbf{b} .$$

(1 point)

(c) $\mathrm{Ran}(A)$ is given by the $\mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2)$ where $\mathbf{a}_j$ are the columns of A. They form a basis $\mathcal{B}$ of $\mathrm{Ran}(A)$. To calculate the orthogonal projection, we can use the Gramian matrix $\mathcal{G}(\mathcal{B})$:

$$\mathcal{G}(\mathcal{B}) = \begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 \end{pmatrix} = A^T A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

(1 point)

Then, we have to solve the same system as in (a) and get $\mathbf{b}|_{\mathrm{Ran}(A)} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$. (1 point)

Alternatively, using Gram-Schmidt:

(c) First normalise:

$$\mathbf{w}_1 := \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Then calculate:

$$\mathbf{v}_1 := \mathbf{a}_2 - \langle \mathbf{a}_2, \mathbf{w}_1 \rangle \mathbf{w}_1 = \mathbf{a}_2 - \frac{1}{\sqrt{6}^2}(2) \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5/3 \\ 5/3 \\ 5/3 \end{pmatrix}$$

(1 point)

Normalise again:

$$\mathbf{w}_2 := \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Now calculate:

$$\mathbf{b}|_{\mathrm{Ran}(A)} = \langle \mathbf{b}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{b}, \mathbf{w}_2 \rangle \mathbf{w}_2 = \frac{2}{3} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \frac{-5}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

(1 point) $\square$

Problem 8: **(4 Points)**

a) Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Show that the LES

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

always has a solution.

Hint: You may use the fact $\mathbb{R}^m = \mathrm{Ran}(A) + \mathrm{Ker}(A^T)$.

b) Let $A \in \mathbb{R}^{m\times n}$. Show that if $\mathrm{Ker}(A) = \{\mathbf{o}\}$, then the solution for the LES $A^T A\mathbf{x} = A^T\mathbf{b}$ is unique.

Solution. (a) The LES has always a solution if and only if $\mathrm{Ran}(A^T A) \supset \mathrm{Ran}(A^T)$:

($\supset$): Choose $y \in \mathrm{Ran}(A^T)$. This means there is a $u \in \mathbb{R}^m$ with $A^T u = y$. Now by the hint, u can be written as $u = u_1 + u_2$ with $u_1 \in \mathrm{Ran}(A)$ and $u_2 \in \mathrm{Ker}(A^T)$. So for u_1 there is an $x \in \mathbb{R}^n$ such that $u_1 = Ax$. Then we get:

$$y = A^T u = A^T u_1 + A^T u_2 = A^T Ax + 0 = A^T Ax$$

Hence $y \in \mathrm{Ran}(A^T A)$. (2 points)

(b) The solution of the LES is unique if and only if $\mathrm{Ker}(A^T A) = \{\mathbf{o}\}$. So choose $x \in \mathrm{Ker}(A^T A)$, which means $A^T Ax = \mathbf{o}$. Hence:

$$0 = \langle A^T Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 .$$

Since the norm is positive definite, we get $Ax = \mathbf{o}$ and $x \in \mathrm{Ker}(A)$. Hence, $x = \mathbf{o}$. (2 points) $\square$