Problem 10.4

$L: V \to W$ linear and bijective. (L^{-1} exists)

(a) Claim: $v_1, \dots, v_k \in V$ lin. independent $\Leftrightarrow$ $Lv_1, \dots, Lv_k \in W$ lin. independent

Proof: ($\Rightarrow$) Let $v_1, \dots, v_k$ lin. independent.

Choose arbitrary $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and then:

$$\alpha_1 L(v_1) + \dots + \alpha_k L(v_k) = 0$$

$$\overset{L^{-1}}{\Longrightarrow} \alpha_1 \cdot v_1 + \dots + \alpha_k v_k = 0 \qquad (L^{-1} \text{ is linear})$$

$$\overset{v_1, \dots, v_k \text{ lin. ind.}}{\Longrightarrow} \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$$

$\Rightarrow$ $Lv_1, \dots, Lv_k$ are lin. independent

($\Leftarrow$) Same idea and proof as before $\square$

(b) Claim: $\dim V = n \Leftrightarrow \dim W = n$

Proof: ($\Rightarrow$) Let $\dim V = n$. This means there is a basis with n elements $B := (v_1, \dots, v_n)$.

$$\overset{(a)}{\Longrightarrow} L(e_1), \dots, L(e_n) \text{ lin. independent.}$$

Let $w \in W$ be arbitrary. Then there is a $v \in V$ with $Lv = w$, since L is bijective.

Since B is a basis of V, there are numbers $\alpha_1, \dots, \alpha_n$ with

$$v = \alpha_1 e_1 + \dots + \alpha_n e_n.$$

Therefore:

$$w = Lv = \alpha_1 L(e_1) + \dots + \alpha_n L(e_n).$$

This means $L(e_1), \dots, L(e_n)$ are also generating.

$\Rightarrow (Le_1, \dots, L(e_n))$ is a basis of W

$\Rightarrow \dim W = n$

($\Leftarrow$) Again, same idea with different names. $\square$

(c) Claim: $A: V \to V$ linear and $B := LAL^{-1}$.

Then: A inj./surj. $\Leftrightarrow$ B inj./surj.

Proof: We use:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "L"'] & V \arrow[d, "L"] \\ W \arrow[r, "B"] & W \end{tikzcd}$$

Injectivity:

($\Rightarrow$) Let A be injective. This means that

$$Av = 0 \quad \Rightarrow \quad v = 0 \qquad (*)$$

Now: $Bw = 0 \Rightarrow LAL^{-1}w = 0$

$\overset{L^{-1}}{\Rightarrow} A(L^{-1}w) = 0$

$\overset{(*)}{\Rightarrow} L^{-1}w = 0 \quad \Rightarrow \quad w = 0$

Hence, B is injective.

($\Leftarrow$) Again, same proof just with renaming.

Surjectivity:

($\Rightarrow$) Let A be surjective. This means that

for all $\tilde{v} \in V$ there exists $v \in \tilde{V}$ with $Av = \tilde{v}$. $(**)$

Choose now $\tilde{w} \in W$ arbitrary. Then for ~~all~~ $L^{-1}\tilde{w}$, there exists $v \in \tilde{V}$ with $Av = L^{-1}\tilde{w}$ by $(**)$.

Choose $w := Lv$ and see that:

$$Bw = LAL^{-1}w = \tilde{w} \quad \Rightarrow \quad B \text{ is surjective}$$

($\Leftarrow$) Again, same argumentation just with different names. $\square$