Exercise 6.5 $T \in B(\mathcal{H})$ self-adjoint, v cyclic vector.

Claim: Eigenvalues have multiplicity one.

Proof: By the spectral theorem (cf. 4.126 Satz (Skript))
T is unitarily equivalent to the multiplication operator.
That means there is a unitary operator $U: \mathcal{H} \to L^2(\mathbb{R}, m)$
with $m(A) := \langle \frac{v}{\|v\|}, \chi_A(T) \frac{v}{\|v\|} \rangle$ for Borel sets A,
and

$$U T U^{-1} = M_{id} \qquad \left((M_{id} f)(t) = t f(t) \right).$$

Moreover $U \frac{v}{\|v\|} = \mathbb{1}$ where $\mathbb{1}(t) = 1 \quad \forall t \in \mathbb{R}$.

If λ is an eigenvalue of M_{id}, then we have
$\chi_{\{\lambda\}}(t) =: g_\lambda(t)$ as an eigenvector $[M_{id} g_\lambda = \lambda g_\lambda]$.

In fact from $(M_{id} - \lambda) f = 0$ follows

$$(t - \lambda) f(t) = 0 \quad m\text{-a.e.}$$

$$\Rightarrow f|_{\mathbb{R} \setminus \{\lambda\}} = 0 \quad m\text{-a.e.}$$

Therefore the eigenspace is spanned by g_λ and the multiplicity is one.