Exercise 6.4

$M_\phi : L^2(\mathbb{R}, \mu) \to L^2(\mathbb{R}, \mu)$ selfadjoint multiplication operator

resolvent set? point spectrum?

Solution: For normal operators hold:

$M_\phi - \lambda$ continuously invertible

$$\iff \exists_{C>0} \quad \|(M_\phi - \lambda)f\|^2 \geq C^2 \|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

(Cf. solution to Exercise 6.2)

$$\iff \exists_{C>0} \quad \int |f(t)|^2 \cdot |\phi(t) - \lambda|^2 \, d\mu(t) \geq C^2 \int |f(t)|^2 \, d\mu(t)$$

$$\iff \exists_{C>0} \quad \int \left(|\phi(t) - \lambda|^2 - C^2 \right) \cdot \underbrace{|f(t)|^2}_{\geq 0} \, d\mu(t) \geq 0$$

$$\iff \exists_{C>0} \quad |\phi(t) - \lambda| \geq C \quad \mu\text{-almost everywhere}$$

Therefore: $\underline{\rho(M_\phi) = \{ \lambda \in \mathbb{C} \mid \exists C > 0 : \ |\phi(t) - \lambda| \geq C \ \mu\text{-a.e} \}}$

For the point spectrum we consider similar things:

Note that:

$M_\phi - \lambda$ is injective

$$\Leftrightarrow \left[(M_\phi - \lambda) f = 0 \quad \Rightarrow f = 0 \right]$$

$$\Leftrightarrow \left[(\phi(t) - \lambda) f(t) = 0 \ \ \mu\text{-a.e} \quad \Rightarrow f(t) = 0 \ \ \mu\text{-a.e.} \right]$$

$$\Leftrightarrow \phi(t) - \lambda \neq 0 \quad \mu\text{-a.e.}$$

$$\Leftrightarrow \mu(\{t \in \mathbb{R} \mid \phi(t) = \lambda\}) = 0$$

Therefore: $\underline{\sigma_p(M_\phi) = \{\lambda \in \mathbb{R} \mid \mu(\{t \in \mathbb{R} \mid \phi(t) = \lambda\}) > 0\}}$

Especially for $\phi(t) = t$:

$$\sigma_p(M_\phi) = \{\lambda \in \mathbb{R} \mid \mu(\{\lambda\}) > 0\}$$