Exercise 6.3 $\quad T \in \mathcal{B}(\mathcal{H})$ self-adjoint. $\quad P(A) := \chi_A(T)$

Claim: $\quad \operatorname{Ran} \chi_{\{\lambda\}}(T) = \operatorname{Kern}(T-\lambda)$

Proof: ($\subset$) Choose $x \in \operatorname{Ran} P(\{\lambda\})$, i.e $\quad P(\{\lambda\})x = x$ (projection!)

For $y \in \mathcal{H}$ arbitrary, one gets:

$$\langle y, (T-\lambda)x\rangle = \langle y, (T-\lambda)P(\{\lambda\})x\rangle$$

$$= \langle y, f(T)x\rangle \quad \text{with} \quad f(t) = (t-\lambda)\chi_{\{\lambda\}}(t)$$

$$\overset{\text{spectral theorem}}{=} \int f(t)\, d\mu_{y,x}(t) \qquad (\text{cf } 4.122 \text{ Satz})$$

$$\left(\text{with } \mu_{y,x}(A) = \langle y, P(A)x\rangle\right)$$

$$= \int_{\{\lambda\}} (t-\lambda)\, d\mu_{y,x}(t)$$

$$= 0 \qquad \Rightarrow \qquad x \in \operatorname{Kern}(T-\lambda)$$

($\supset$) Choose $x \in \operatorname{Kern}(T-\lambda)$, i.e. $Tx = \lambda x$ and therefore:

$$x = \underbrace{\chi_{\{\lambda\}}(\lambda)}_{1} x = \chi_{\{\lambda\}}(T)x = Px \qquad \Rightarrow \qquad x \in \operatorname{Ran} P$$

$\square$

$[f(T)x = f(\lambda)x]$
holds for polynomials
and with 4.117 Satz
for all bounded Borel functions!