Exercise 6.2

Claim: $T \in \mathcal{B}(\mathcal{H})$ normal, Then:

$$\lambda \in \sigma(T) \iff \exists (x_n) \subseteq \mathcal{H} \text{ with } \|x_n\| = 1 \text{ and } \|(T-\lambda)x_n\| \xrightarrow{n\to\infty} 0$$

Proof: We need the following general property:

Proposition: $T \in \mathcal{B}(\mathcal{H})$ and normal. Then:

T continuously invertible $\iff \|Tx\| \geq C \cdot \|x\| \quad \forall x \in \mathcal{H}$ for a $C > 0$.

Proof: $(\Rightarrow)$ $\|x\| \leq \|T^{-1}\| \, \|Tx\| \Rightarrow \|Tx\| \geq \underbrace{\frac{1}{\|T^{-1}\|}}_{>0} \cdot \|x\|$ ✓

$(\Leftarrow)$ Proof like in A6.1 / Exercise 6.1.

- T is injective ($Tx = 0 \Rightarrow C\|x\| = 0 \Rightarrow x = 0$)
- $\mathrm{Ran}(T)$ is closed ($\|x_n - x_m\| \leq \|T^{-1}\| \, \|Tx_n - Tx_m\|$

  So for each sequence $(x_n) \subseteq \mathcal{H}$ with $Tx_n \to y \in \mathcal{H}$

  we know that (x_n) has to be Cauchy $\Rightarrow x_n \to x \in \mathcal{H}$

  Since T is continuous $\Rightarrow Tx_n \to Tx = y \Rightarrow y \in \mathrm{Ran}(T)$)
- $\overline{\mathrm{Ran}(T)} = \mathcal{H}$. Here we use the normality of T:

  Since $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = \langle x, TT^*x \rangle$

  $= \langle T^*x, T^*x \rangle = \|T^*x\|^2$

  we know $\mathrm{Kern}(T) = \mathrm{Kern}(T^*)$

Therefore: $\operatorname{Ran}(T) = \overline{\operatorname{Ran}(T)} = \operatorname{Kern}(T^*)^\perp = \operatorname{Kern}(T)^\perp = \{0\}^\perp = \mathcal{H}$.

This proves the Proposition $\boxed{P}$

The rest is now easy to show:

$$\lambda \notin \sigma(T) \overset{\text{Prop}}{\iff} \exists_{C>0} \ \forall_{x \in \mathcal{H}} \quad \|Tx\| \geq C\|x\|$$

$$\iff \forall_{\substack{(x_n) \subseteq \mathcal{H} \\ \|x_n\|=1}} \quad \|(T-\lambda)x_n\| \not\to 0$$

$$\iff \neg \Big(\exists_{\substack{(x_n) \subseteq \mathcal{H} \\ \|x_n\|=1}} \quad \|(T-\lambda)x_n\| \to 0 \Big)$$

Or otherwise:

$$\lambda \in \sigma(T) \iff \exists_{\substack{(x_n) \subseteq \mathcal{H} \\ \|x_n\|=1}} \quad \|(T-\lambda)x_n\| \to 0 \qquad \square$$