Aufgabe 6.1

X reflexiv Banach space, $T \in \mathcal{B}(X)$

Claim: $\sigma(T) \subseteq \overline{W(T)}$ with $W(T) := \{x'(Tx) \in \mathbb{C} \mid \|x\| = \|x'\| = 1, x'(x) = 1\}$

Proof: Choose $\lambda \notin \overline{W(T)}$. Then $d := \operatorname{dist}(\lambda, W(T)) > 0$.

Let $x \in X$ with $\|x\| = 1$ be arbitrary.

$\overset{\text{Hahn-Banach}}{\Longrightarrow}$ There is a $x' \in X'$ with $\|x'\| = 1$ and $x'(x) = \|x\| = 1$.

$$\Rightarrow 0 < d \le |\underset{\lambda \cdot \underbrace{x'(x)}_{=1}}{\lambda} - \underset{\in W(T)}{x'(Tx)}| = |x'((\lambda - T)x)| \le \underbrace{\|x'\|}_{=1} \cdot \|(T-\lambda)x\|$$

And therefore: $\|(T-\lambda)x\| \ge d \cdot \|x\| \quad \forall x \in X$.

It follows $(T-\lambda)$ injective $\left((T-\lambda)x = 0 \Rightarrow d\|x\| = 0 \Rightarrow x = 0 \right)$

$(T-\lambda)^{-1} : \operatorname{Ran}(T-\lambda) \to X$ is continuous.

$$\left(\|(T-\lambda)^{-1} y\| \le \frac{1}{d} \|y\| \right)$$

Now we show two things:

(1) $\operatorname{Ran}(T-\lambda)$ closed

(2) $\overline{\operatorname{Ran}(T-\lambda)} = X$

Proof of (1): Choose sequence $(x_n) \subseteq X$ with $(T-\lambda)x_n \to y \in X$.

Since $\|x_n - x_m\| = \|(T-\lambda)^{-1}(T-\lambda)(x_n - x_m)\| \le \|(T-\lambda)^{-1}\| \cdot \|(T-\lambda)x_n - (T-\lambda)x_m\|$

(x_n) is a C.S. in X. $\overset{X \text{ B.S}}{\Longrightarrow}$ $x_n \to x \in X$

$\overset{(T-\lambda) \text{ cont.}}{\Longrightarrow}$ $(T-\lambda)x_n \to (T-\lambda)x = y \Rightarrow y \in \operatorname{Ran}(T-\lambda) \Rightarrow \operatorname{Ran}$ closed

Proof of (2)

Assume $\mathrm{Kern}((T-\lambda)') \neq \{0\}$. Then there is $x' \in \mathrm{Kern}((T-\lambda)')$ with $\|x'\| = 1$.

By James we have an $x \in X$ with $x'(x) = 1$, ($\|x\| = 1$).

$$\Rightarrow \quad |\lambda - \underbrace{x'(Tx)}_{\in W(T)}| = |x'((T-\lambda)x)| = |(T-\lambda)'(x')(x)| = 0$$

$$\Rightarrow \quad \lambda \in \overline{W(T)}.$$

By contraposition: $\underline{\lambda \notin \overline{W(T)} \Rightarrow \mathrm{Kern}((T-\lambda)') = \{0\}}$.

By Satz 4.32 (Skript):

$$\overline{\mathrm{Ran}(T-\lambda)} = \mathrm{Kern}((T-\lambda)')_{\perp}$$

$$= \{0\}_{\perp} = X$$

With (1), (2): $\mathrm{Ran}(T-\lambda) = X$, i.e. $T-\lambda$ surjective.

In combination with the injectivity we get: $\lambda \notin \sigma(T)$.

⌜So $\lambda \notin \overline{W(T)} \Rightarrow \lambda \notin \sigma(T)$

and $\lambda \in \sigma(T) \Rightarrow \lambda \in \overline{W(T)}$⌟ $\square$