Exercise 7 $T \in \mathcal{B}(\mathcal{H})$ and $T \geq 0$

Claim: T bijective $\Leftrightarrow \exists \varepsilon > 0: T - \varepsilon \geq 0$ (*)

Proof: By spectral mapping theorem: $\sigma(f(T)) = f(\sigma(T))$

Now: $f(x) = x - \varepsilon$, $f(T) = T - \varepsilon$

$$\Rightarrow \sigma(T - \varepsilon) = \sigma(T) - \varepsilon$$

Therefore:

$$\sigma(T-\varepsilon) \subseteq [0,\infty) \Leftrightarrow \sigma(T) \subseteq [\varepsilon, \infty)$$

$$\Leftrightarrow 0 \notin \sigma(T)$$

$$\Leftrightarrow T \text{ invertible} \qquad \square$$

Claim: For operator $S \in \mathcal{B}(\mathcal{H})$ we have:

$$S \text{ bijective} \Leftrightarrow \exists \varepsilon > 0: \; S^*S - \varepsilon \geq 0 \text{ and } SS^* - \varepsilon \geq 0$$

Proof: ($\Rightarrow$) S bijective $\Rightarrow S^*, SS^*, S^*S$ bijective

$$\overset{(*)}{\Rightarrow} \exists \varepsilon_1 > 0, \exists \varepsilon_2 > 0 \text{ with } S^*S - \varepsilon_1 \geq 0, \; SS^* - \varepsilon_2 \geq 0$$

Choose $\varepsilon := \min(\varepsilon_1, \varepsilon_2) \Rightarrow S^*S - \varepsilon \geq 0$ and $SS^* - \varepsilon \geq 0$.

($\Leftarrow$) With (*) S^*S and SS^* bijective, i.e.

$$\left.\begin{aligned} \{0\} &= \mathrm{Kern}(S^*S) \supseteq \mathrm{Kern}(S) \\ \mathcal{H} &= \mathrm{Bild}(SS^*) \subseteq \mathrm{Bild}(S) \end{aligned}\right\} \Rightarrow S \text{ bijective} \qquad \square$$