Exercise 6

$A, B \in \mathcal{B}(\mathcal{H})$, $A \geq 0$, AB self-adjoint.

Claim: $AB \leq r(B) A$

Proof: $|\langle ABx, x\rangle|^2 = |\langle \sqrt{A}Bx, \sqrt{A}x\rangle|^2$

$$\leq \langle \sqrt{A}Bx, \sqrt{A}Bx\rangle \cdot \langle \sqrt{A}x, \sqrt{A}x\rangle$$

$$= \langle B^* \sqrt{A}^2 B x, x\rangle \langle Ax, x\rangle \quad = (*)$$

Note: $(AB)^* = AB \Rightarrow B^* A^* = AB$

$\Rightarrow B^* A = AB$

$(*) = \langle AB^2 x, x\rangle \langle Ax, x\rangle$

By induction one shows: $|\langle ABx, x\rangle|^{2^n} \leq \langle AB^{2^n} x, x\rangle \langle Ax, x\rangle^{2^n - 1}$

So we have: $|\langle ABx, x\rangle|^{2^n} \leq \|A\| \cdot \|B^{2^n}\| \cdot \|x\|^2 \langle Ax, x\rangle^{2^n - 1}$

Therefore: $|\langle ABx, x\rangle| \leq \lim_{n \to \infty} \sqrt[2^n]{\|A\| \cdot \|B^{2^n}\| \cdot \|x\|^2} \langle Ax, x\rangle^{1 - \frac{1}{2^n}}$

$$= r(B) \langle Ax, x\rangle \qquad \square$$