Exercise 5

$\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces.

Claim: $\forall A_1, \dots, A_n \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\forall \alpha_1, \dots, \alpha_n \in \mathbb{C}$:

$$\left|\sum_{i=1}^n \alpha_i A_i\right|^2 \leq \left(\sum_{i=1}^n |\alpha_i|^2\right)\left(\sum_{i=1}^n |A_i|^2\right)$$

Proof: By definition: $|A| = \sqrt{A^*A}$ and therefore $|A|^2 = A^*A$

$$\left\| \sum \alpha_i A_i x \right\|^2 \underset{\Delta\text{-inequ.}}{\leq} \left(\sum_{i=1}^n |\alpha_i| \, \|A_i x\| \right)^2 = (*)$$

Now use: $\sum \overline{x_i} y_i = \langle x, y\rangle_{\mathbb{C}^n} \overset{\text{C.S.}}{\leq} \|x\| \cdot \|y\|$

$$= \sqrt{\sum x_i^2} \cdot \sqrt{\sum y_i^2}$$

$$(*) \leq \left(\sum |\alpha_i|^2\right) \cdot \left(\sum \|A_i x\|^2\right)$$

So we have shown:

$$\left\langle \left(\sum \alpha_i A_i\right)^* \left(\sum \alpha_i A_i\right) x, x \right\rangle \leq \left(\sum |\alpha_i|^2\right) \cdot \sum \langle A_i^* A_i x, x\rangle$$

$$\leq \left(\sum |\alpha_i|^2\right) \left\langle \sum A_i^* A_i x, x \right\rangle \quad \forall x \in \mathcal{H}.$$

With $|A_i|^2 = A_i^* A_i$ this is the claim. (of course you need the <u>polarisation identity!</u>) $\square$