Exercise 4 $\quad A, B \geq 0, \; AB = BA$

Claim: $\quad A \leq B \iff \sqrt{A} \leq \sqrt{B}$

$$\frac{A+B}{2} \geq \sqrt{AB}$$

$$\sqrt{\frac{A+B}{2}} \geq \frac{\sqrt{A}+\sqrt{B}}{2}$$

Proof: By spectral theorem:

$AB = BA \implies \sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$ and $AB = \sqrt{A}\, B \sqrt{A} \geq 0$ (cf. 4.51 Satz)

Since $(\sqrt{A}\sqrt{B})^2 = AB$ and square root is unique, we have

$$\sqrt{AB} = \sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A} \qquad (\text{since } AB \geq 0\ !)$$

(see above)

(1) Assume $A \leq B \implies B - A \geq 0$

$$\implies B^2 - A^2 = (B-A)(B+A) \geq 0 \qquad (AB = BA\ !)$$

$$\implies B^3 - A^3 = (B-A)(B^2 + AB + A^2) \geq 0$$

By induction $B^n - A^n = (B-A)(B^{n-1} + \dots + A^{n-1}) \geq 0$

By Weierstraß: $\sqrt{B} - \sqrt{A} \geq 0$. ✓

(2) Assume $\sqrt{A} \leq \sqrt{B} \implies B - A = (\sqrt{B} - \sqrt{A})(\sqrt{B} + \sqrt{A}) \geq 0$

$$\implies A \leq B \qquad ✓$$

(3) $(A-B)^2 \geq 0 \implies \left(\frac{A+B}{2}\right)^2 \geq AB \implies \frac{A+B}{2} \geq \sqrt{AB}$

(4) $\frac{A+B}{2} \geq \sqrt{AB}$

$$\Rightarrow \frac{A+B}{2} + \frac{A+B}{2} \geq \frac{A+B}{2} + \sqrt{AB}$$

$$\Rightarrow \frac{A+B}{2} \geq \frac{A+B+2\sqrt{A}\sqrt{B}}{4}$$

$$\Rightarrow \frac{A+B}{2} \geq \left(\frac{\sqrt{A}+\sqrt{B}}{2}\right)^2$$

$$\overset{(1)}{\Rightarrow} \sqrt{\frac{A+B}{2}} \geq \frac{\sqrt{A}+\sqrt{B}}{2} \quad \checkmark$$