Exercise 2

Claim: $\sigma(AB)\setminus\{0\} = \sigma(BA)\setminus\{0\}$

Proof: (1) Show: $I-AB$ continuously invertible

$\Longleftrightarrow I-BA$ continuously invertible

Proof: $(\Leftarrow)$ Formally:

$$(I-AB)^{-1} = \sum_{k=0}^{\infty}(AB)^k \quad \text{if possible!}$$

$$= I + \sum_{k=0}^{\infty}\underbrace{(AB)^{k+1}}_{A(BA)^k B}$$

$$= I + A(I-BA)^{-1}B$$

So set $Z := I + A(I-BA)^{-1}B$

exists by assumption!

and calculate:

$$Z(I-AB) = (I-AB) + A(I-BA)^{-1}B(I-AB)$$

$$= (I-AB) + A\left[\underbrace{(I-BA)^{-1}(I-BA)}_{I}B\right]$$

$$= I \quad \checkmark$$

$$(I-AB)Z = (I-AB) + (I-AB)A(I-BA)^{-1}B$$

$$= I \quad \checkmark$$

Therefore $I - AB$ cont. invertible with inverse z!

($\Rightarrow$) Same proof with $A \Leftrightarrow B$.

(2) With (1) we can calculate for $\lambda \neq 0$:

$$AB - \lambda = -\lambda \underbrace{\left(I - \frac{1}{\lambda} AB \right)}_{\text{cont. invertible}} \overset{(1)}{\Longleftrightarrow} \left(I - \frac{1}{\lambda} BA \right) \text{ cont. invertible}$$

So: $AB - \lambda$ cont. invertible $\Leftrightarrow$ $BA - \lambda$ cont. invertible

That means: $\underline{\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}}$. $\square$

Example for $\lambda = 0$:

$$A : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$$
$$B : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$$

on $(\ell^p, \|\cdot\|_p)$

Then: $AB = I \Rightarrow 0 \notin \sigma(AB)$

$BA e_1 = 0 \Rightarrow 0 \in \sigma(BA)$ $\lightning$

$e_1 = (1, 0, \dots)$