Exercise 1

<u>Claim:</u> $B_+ = \{ T \in \mathcal{B}(\mathcal{H}) \mid T \geq 0 \,,\, \|T\| \leq 1 \}$ convex

and every orthogonal projection is extremal.

<u>Proof:</u>

(1) B_+ is convex:

Choose $T, S \in B_+$ and $\lambda \in [0,1]$:

$$\langle x, (\lambda T + (1-\lambda) S) x \rangle \geq 0 \quad ✓$$

$$\| \lambda T + (1-\lambda) S \| \leq \lambda \|T\| + (1-\lambda) \|S\|$$

$$\leq \lambda + (1-\lambda) \leq 1 \quad ✓$$

(2) Choose $x \in \mathrm{Ran}(P)$:

$$\|x\| = \|Px\| = \| \lambda Tx + (1-\lambda) Sx \|$$

$$\leq \lambda \|Tx\| + (1-\lambda) \|Sx\|$$

$$\leq (\lambda \|T\| + (1-\lambda) \|S\|) \cdot \|x\| \overset{(1)}{\leq} 1 \cdot \|x\|$$

$\Rightarrow$ Every step $\overset{!}{=}$

In particular: $\| Tx_1 + Sx_2 \| = \|Tx_1\| + \|Sx_2\|$ $\qquad x_1 := \lambda x$, $x_2 := (1-\lambda) x$

$$\Rightarrow \|Tx_1\| \cdot \|Sx_2\| = \mathrm{Re} \langle Tx_1, Sx_2 \rangle \leq \|Tx_1\| \cdot \|Sx_2\|$$

equality if and only if Tx, Sx colinear!

$\Rightarrow$ $x = Px = \lambda \cdot Tx + (1-\lambda) Sx$ is also in the 1-dim subspace $U := \mathrm{linspan}\{x, Tx, Sx\} = \mathrm{linspan}\{x\}$ $\quad (**)$

Choose $x \in \mathrm{Kern}(P)$:

$$0 = \langle x, Px \rangle = \underbrace{\lambda}_{>0} \underbrace{\langle x, Tx \rangle}_{\geq 0} + \underbrace{(1-\lambda)}_{>0} \underbrace{\langle x, Sx \rangle}_{\geq 0}$$

$$\Rightarrow \langle x, Tx \rangle = 0 \ \wedge \ \langle x, Sx \rangle = 0$$

By polarization: $\langle x, Ty \rangle = \frac{1}{4}\left(\langle x+y, T(x+y) \rangle \ \ldots \ \right)$

$$= \underline{0} \quad \text{for all } x, y \in \mathrm{Kern}(P) \quad (***)$$

We know: $\mathcal{H} = \mathrm{Ker}(P) \oplus \mathrm{Ran}(P) \ni z_1 + z_2 =: z$, $x \in \mathrm{Kern}(P)$

Then $\langle z, Tx \rangle = \underbrace{\langle Tz_1, x \rangle}_{\overset{(***)}{=} 0} + \langle Tz_2, x \rangle$

$$= \langle \mu z_2, x \rangle \qquad \text{for } \mu \in \mathbb{C} \text{ by } (**)$$

$$= 0 \qquad \text{since } \mathrm{Kern}(P)^\perp = \mathrm{Ran}(P)$$

Therefore: $Tx = 0$ for all $x \in \mathrm{Kern}(P)$

and analogously $Sx = 0$ " .

So we have: $T^2 = T$ since $\|T\| = \|S\| = 1$

$S^2 = S$ (Remember: $1 \leq \lambda \underbrace{\|T\|}_{\leq 1} + (1-\lambda) \underbrace{\|S\|}_{\leq 1}$)

and $Tx = x$
$Sx = x$ for all $x \in \mathrm{Ran}(P)$

(Note $x = \lambda Tx + (1-\lambda) Sx$

and Tx, x, Sx colinear!)

$\Rightarrow Tx = x \wedge Sx = x$

$$\text{and} \quad \begin{aligned} Tx &= 0 \\ Sx &= 0 \end{aligned} \qquad \text{for all } x \in \mathrm{Ker}(P)$$

$$\text{So} \quad \left.\begin{aligned} \mathrm{Ker}(T) &= \mathrm{Ker}(S) = \mathrm{Ker}(P) \\ \mathrm{Ran}(T) &= \mathrm{Ran}(S) = \mathrm{Ran}(P) \end{aligned}\right\} \Rightarrow T = S = P \qquad \square$$