Exercise 5

$$T: C([0,1]) \longrightarrow C([0,1])$$

$$(Tx)(t) = \int_0^t x(\tilde{t})\, d\tilde{t}$$

What is the spectrum?

$T$ is bounded: $\|Tx\|_\infty \leq \|x\|_\infty$

And the spectral radius is given by $r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$

Let $\|x\|_\infty = 1$ and calculate:

$$|Tx(t)| \leq \int_0^t |x(\tilde{t})|\, d\tilde{t} \leq t$$

$$|T^2x(t)| \leq \int_0^t |Tx(\tilde{t})|\, d\tilde{t} \leq \frac{1}{2} t^2$$

inductively: $|(T^n x)(t)| \leq \frac{1}{n!} t^n$

And hence the operator norm

$$\|T^n\| = \sup_{\|x\|_\infty = 1} \sup_{t\in[0,1]} |T^n x(t)| \leq \frac{1}{n!}$$

$$\Rightarrow r(t) \leq \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0 \qquad \left(\frac{1}{n!} \overset{\text{Stirling}}{\leq} \frac{1}{\sqrt{2n}} \frac{1}{n^{n+\frac{1}{2}}} e^n\right)$$

$$\Rightarrow \sigma(T) \subseteq \{0\} \quad \overset{\sigma \neq \emptyset}{\Longrightarrow} \quad \underline{\sigma(T) = \{0\}}$$