Exercise 4

Let T be the left shift on $\ell^1(\mathbb{N})$:

$$T : \ell^1(\mathbb{N}) \longrightarrow \ell^1(\mathbb{N})$$

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

and S the right shift on $\ell^\infty(\mathbb{N})$

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

(a) Claim: $\sigma(T) = \sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$

Proof: Note that $\ell^\infty(\mathbb{N}) \cong \ell^1(\mathbb{N})'$ via the identification:

$$\ell^1(\mathbb{N})' \ni y' \mapsto y \in \ell^\infty(\mathbb{N})$$

$$y'(z) = \sum_{n=0}^{\infty} y_n z_n \qquad \forall z \in \ell^1(\mathbb{N})$$

Then: $(T'y')(x) = y'(Tx) = \sum_{n=0}^{\infty} y_n (Tx)_n = \sum_{n=0}^{\infty} y_n x_{n+1}$

$$= \sum_{n=0}^{\infty} \tilde{y}_{n+1} x_{n+1} \qquad \text{with } \tilde{y}_n := y_{n-1}$$

$$\tilde{y} = (0, y_1, y_2, \dots)$$

That means $T' \subseteq S$.

With Exercise 3 we get $\sigma(T) = \sigma(T') = \sigma(S)$.

Moreover: For $|\lambda| \leq 1$ we define:

$$x_\lambda := (1, \lambda, \lambda^2, \dots) \in \ell^1(\mathbb{N})$$

Then: $Tx_\lambda = (\lambda, \lambda^2, \dots) = \lambda(1, \lambda, \lambda^2, \dots) = \lambda x_\lambda$

$\Rightarrow \lambda \in \sigma_p(T) \Rightarrow \underbrace{\overline{\mathbb{D}}}_{= \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}} \subseteq \sigma(T)$ since the spectrum is closed.

Since $\|T\| = 1$ $\left(\|Tx\|_{\ell^1} = \sum_{k=2}^{\infty} |x_k| \leq \|x\|_{\ell^1} \right)$,

we know $\sigma(T) \subseteq \overline{\mathbb{D}} \Rightarrow \overline{\mathbb{D}} = \sigma(T) = \sigma(S)$ $\square$

<u>(b)</u>

We have already shown $\mathbb{D} \subseteq \sigma_p(T)$

Choose $\lambda \in \overline{\mathbb{D}}$

$y \in \operatorname{Kern}(S - \lambda) \Rightarrow Sy = \lambda y \Rightarrow$
$$\begin{aligned} 0 &= \lambda y_1 \\ y_1 &= \lambda y_2 \\ y_2 &= \lambda y_3 \\ &\vdots \end{aligned}$$

$\Rightarrow y = 0 \vee \lambda = 0 \overset{S \text{ inj.}}{\Rightarrow} y = 0 \Rightarrow$ <u>$S - \lambda$ inject.</u>

$\Rightarrow \sigma_p(S) = \emptyset$

For $\lambda \in \mathbb{D}$ assume $\mathrm{Ran}(S-\lambda)$ dense in ℓ^∞:

$$\ell^\infty = \overline{\mathrm{Ran}(S-\lambda)} \overset{\text{Ex3}}{\subseteq} \mathrm{Kern}(T-\lambda)^\perp$$

$$\Rightarrow \mathrm{Kern}(T-\lambda) = \{0\} \quad ↯$$