Exercise 3 $\quad X, Y$ Banach spaces

(a) To know: (Hahn-Banach)

$$\|x\| = \sup_{\|x'\| \leq 1} |x'(x)| \quad \text{for all } x \in X.$$

Then it follows:

$(x_n) \subseteq X$ Cauchy sequence

$$\Longleftrightarrow \forall_{\varepsilon > 0} \; \exists_{N \in \mathbb{N}} \; \forall_{n,m \geq N} \quad \underbrace{\|x_n - x_m\|}_{= \sup_{\|x'\| \leq 1} |x'(x_n - x_m)|} < \varepsilon$$

$$\Longleftrightarrow \forall_{\varepsilon > 0} \; \exists_{N \in \mathbb{N}} \; \forall_{n,m \geq N} \; \forall_{\substack{x' \in X' \\ \|x'\| \leq 1}} \quad |x'(x_n) - x'(x_m)| < \varepsilon$$

(b) $T \in \mathcal{B}(X, Y)$, $T' \in \mathcal{B}(Y', X')$ defined by $T'(y')(x) = y'(Tx)$

Claim: T cont. invertible $\Longleftrightarrow$ T' cont. invertible

Proof: Use Satz 4.32 (Skript):

$$\overline{\operatorname{Ran}(T)} = \operatorname{Kern}(T')_{\perp} \qquad (1)$$

This implies $\overline{\operatorname{Ran}(T')} = \operatorname{Kern}(T'')_{\perp} \overset{\text{Extension}}{\subseteq} \operatorname{Kern}(T)^{\perp} \quad (2)$

($\Rightarrow$) T invertible with inverse T^{-1}. Then

$$TT^{-1} = T^{-1}T = I$$

$$\Rightarrow (T^{-1})' T' = T'(T^{-1})' = I' = I$$

$\Rightarrow T'$ is cont. invertible with inverse $(T^{-1})'$.

($\Leftarrow$) Assume T' cont. invertible. Then:

$$\operatorname{Kern}(T') = \{0\} \quad \wedge \quad \operatorname{Ran}(T') = X'$$

$$\overset{(1),(2)}{\Rightarrow} \{0\}_\perp = \overline{\operatorname{Ran}(T)} \quad \text{and} \quad \operatorname{Kern}(T)^\perp = X'$$

$$\Rightarrow \overline{\operatorname{Ran}(T)} = \underbrace{\{y \in Y \mid y'(y) = 0 \quad \forall y' \in \{0\}\}}_{= Y}$$

$$\text{and} \quad X' = \{x' \in X' \mid x'(x) = 0 \quad \forall x \in \operatorname{Kern}(T)\}$$

$$\underset{\text{(Hahn-Banach)}}{\Longrightarrow} \operatorname{Kern}(T) = \{0\}$$

It remains to show:

Ran(T) closed:

Choose a sequence $Tx_n \to y \in Y$. Then for all $x' \in X'$ with $\|x'\| \leq 1$ holds:

$$|x'(x_n - x_m)| = |((T')^{-1} \underbrace{T' x' (x_n - x_m)}_{x'(T(x_n-x_m))})|$$

$$\leq \|(T')^{-1}\| \cdot \|x'(T(x_n - x_m))\|$$

$$\leq \|(T')^{-1}\| \cdot \underbrace{\|Tx_n - Tx_m\|}_{\longrightarrow 0} \qquad \forall x' \in X' \text{ with } \|x'\| \leq 1$$

$\overset{(a)}{\Longrightarrow}$ (x_n) C.S. in X with limit $x \in X$ (Banach space)

Since T is continuous, we have $y = Tx$

$\Longrightarrow$ $\mathrm{Ran}(T)$ is closed $\Longrightarrow$ T bijektiv

$\overset{\text{open mapping theorem}}{\Longrightarrow}$ T cont. invertible $\square$

(c) Claim: $\sigma(T) = \sigma(T')$ for $X = Y$

Proof: $\lambda \in \rho(T) \Longleftrightarrow T - \lambda I$ cont. invertible

$\overset{(b)}{\Longleftrightarrow}$ $T' - \lambda I$ cont. invertible

$\Longleftrightarrow$ $\lambda \in \rho(T')$

Therefore $\sigma(T) = \sigma(T')$ $\square$

Note: For Hilbert space: $\sigma(T^*) = \{\overline{\lambda} \in \mathbb{K} \mid \lambda \in \sigma(T)\}$