Exercise 1

$$\mathcal{D}: \left(C^1([0,1]), \|\cdot\|_\infty\right) \to \left(C^0([0,1]), \|\cdot\|_\infty\right)$$
$$f \mapsto f'$$

Claim: $\mathcal{D}$ is unbounded but a closed operator

Solution:

(1) $\|\mathcal{D}\| \geq \|\mathcal{D}f\|_\infty$ for $f \in C^1$ with $\|f\|_\infty = 1$

Choose $f_n = t^n$

$$\Rightarrow f_n' = n \cdot t^{n-1}$$

$$\|\mathcal{D}\| \geq \|\mathcal{D}f_n\|_\infty = n \cdot \sup_{t \in [0,1]} t^{n-1} = n \xrightarrow{n \to \infty} \infty$$

(2) Closeness:

Criterion for closeness:

For every sequence $(x_n) \subseteq \mathrm{Dom}(A)$ with $\lim_{n\to\infty} x_n = x$ and $Ax_n \to y$ follows $x \in \mathrm{Dom}(A)$ and $Ax = y$. Then A is closed!

Choose a sequence $(f_n) \subseteq C^1$ with $\|f_n - f\|_\infty \to 0$ and $\|f_n' - g\|_\infty \to 0$, for $f \in C^1$, $g \in C^0$.

To show: $f' = g$ (see Analysis I)

For $x, y \in [0,1]$, $m, n \in \mathbb{N}$ we have:

$$\left| \frac{f_m(y) - f_m(x)}{y - x} - \frac{f_n(y) - f_n(x)}{y - x} \right|$$

$$\overset{\text{Mittelwertsatz}}{=} \left| (f_m - f_n)'(\xi_{y,x}) \right|$$

$$\leq \underbrace{\| f_m' - f_n' \|_\infty}_{\text{indepent of } x, y} \xrightarrow{m,n \to \infty} 0$$

So: $\forall_{\varepsilon > 0} \; \exists_{N \in \mathbb{N}} \; \forall_{m,n \geq N} \; \forall_{x,y \in [0,1]} \left| \underbrace{\frac{f_m(y) - f_m(x)}{y - x}}_{\substack{m \to \infty \\ \frac{f(y) - f(x)}{y - x}}} - \frac{f_n(y) - f_n(x)}{y - x} \right| < \varepsilon$

So in the limit $y \to x$ we get:

$$\forall_{\varepsilon > 0} \; \exists_{N \in \mathbb{N}} \; \forall_{m,n \geq N} \; \forall_{x,y \in [0,1]} \; |f'(x) - f_n'(x)| < \varepsilon$$

Or in other words:

$$\| f_n' - f' \|_\infty \xrightarrow{n \to \infty} 0$$

Since the limit is unique, we get $\underline{f' = g}$ $\square$