A.7.2

(a) $$\int_{\partial B_2(0)} \frac{e^z}{z(1-z^2)}\,dz = 2\pi i\left[Res_0 + Res_1 + Res_{-1}\right] = \underline{2\pi i\left(1-\frac{e^{-1}}{2}-\frac{e}{2}\right)}$$

mit $Res_0 = \frac{e^0}{1-0^2} = 1$, $Res_1 = \frac{e^1}{(-1)\cdot(1+1)} = \frac{e}{2}$, $Res_{-1} = \frac{e^{-1}}{(-1)(1+1)} = \frac{e^{-1}}{2}$

(b) $$\int_0^{2\pi} \frac{\cos(3t)}{4-3\cos(t)}\,dt = \int_0^{2\pi} \frac{\frac{1}{2}(e^{3it}+e^{-3it})}{4-\frac{3}{2}(e^{it}+e^{-it})}\cdot\frac{ie^{it}}{ie^{it}}\,dt =$$

$$= \frac{1}{2}\oint_{\partial B_1(0)} \frac{z^3+z^{-3}}{4-\frac{3}{2}(z+z^{-1})}\,\frac{1}{iz}\,dz = \frac{1}{2i}\oint_{\partial B_1(0)} \frac{z^6+1}{4z^4-\frac{3}{2}z^5-\frac{3}{2}z^3}\,dz$$

$$= \frac{1}{2i}\oint_{\partial B_1(0)} \frac{z^6+1}{\frac{3}{2}z^3\cdot(-1)\cdot\left(z-\frac{1}{3}(4-\sqrt{7})\right)\left(z-\frac{1}{3}(4+\sqrt{7})\right)}\,dz$$

$$= \pi\cdot\left(Res_0 + Res_{\frac{1}{3}(4-\sqrt{7})}\right) = \underline{\pi\cdot\left(-\frac{110}{27}+\frac{296}{189}\sqrt{7}\right)}$$

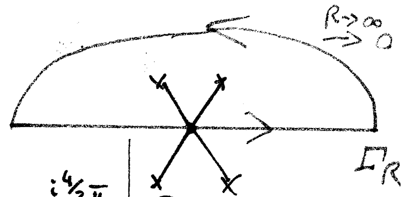
mit $$Res_0 = \frac{1}{2}\cdot\frac{\partial^2}{\partial z^2}\Big|_{z=0}\left[z^3\frac{z^6+1}{4z^4-\frac{3}{2}z^5-\frac{3}{2}z^3}\right]$$ Pol 3. Ordnung

$$= \frac{1}{2}\cdot\left(-\frac{220}{27}\right)$$, Quotientenregel + Produktregel

$$= -\frac{200}{2\cdot 27} = \underline{-\frac{110}{27}}$$

und $$Res_{\frac{1}{3}(4-\sqrt{7})} = \frac{1+\left(\frac{1}{3}(4-\sqrt{7})\right)^6}{-\frac{3}{2}\left(\frac{1}{3}(4-\sqrt{7})\right)^3\cdot\left(\frac{1}{3}(4-\sqrt{7})-\frac{1}{3}(4+\sqrt{7})\right)} = \underline{\frac{296}{27\cdot 7}\cdot\sqrt{7}}$$

(c) $$\int_{-\infty}^{\infty} \frac{1}{x^4+x^2+1}\,dx = \lim_{R\to\infty}\oint_{\Gamma_R} \frac{1}{z^4+z^2+1}\,dz$$



NST $= e^{i\frac{1}{3}\pi}, e^{i\frac{4}{3}\pi}, e^{-i\frac{1}{3}\pi}, e^{i\frac{2}{3}\pi}$

Einfache Pole

Nicht in oberer Halbebene

$$= 2\pi i\cdot\left(Res_{e^{i\frac{\pi}{3}}} + Res_{e^{i\frac{2}{3}\pi}}\right)$$

$$= \underline{\frac{\pi}{\sqrt{3}}}$$

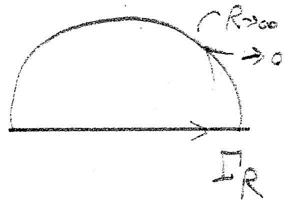
mit $$Res_{e^{i\pi/3}} = \frac{1}{4e^{i\pi}+2e^{i\pi/3}} = \underline{\frac{1}{2e^{i\pi/3}-4}}$$

und $\mathrm{Res}_{e^{i\frac{2}{3}\pi}} = \dfrac{1}{4e^{2\pi i} + 2e^{i\frac{2}{3}\pi}} = \underline{\dfrac{1}{2e^{i\frac{2}{3}\pi}+4}}$

<u>In der Summe:</u> $\mathrm{Res}_{e^{i\frac{\pi}{3}}} + \mathrm{Res}_{e^{i\frac{2}{3}\pi}} = \dfrac{2\left(e^{i\,2/3\,\pi} + e^{i\pi/3}\right)}{4e^{\pi i} + 8\left(e^{i\pi/3} - e^{i\,2/3\,\pi}\right) - 16}$

$$= \frac{2\cdot i\,\frac{1}{2}\sqrt{3}\cdot 2}{8\cdot\frac{1}{2}2 - 20} = \frac{2i\cdot 3}{-12\cdot\sqrt{3}} = \underline{\frac{1}{2i}\cdot\frac{1}{\sqrt{3}}}$$

<u>(d)</u> $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2}\,dx = \cancel{\lim_{R\to\infty}\int_{\Gamma_R} \frac{\cos(z)}{(1+z^2)(1+z^2)}\,dz}$



Funktion fällt schnell genug ab im Unendlichen.

$$= \mathrm{Re}\left(\lim_{R\to\infty}\int_{\Gamma_R} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz\right)$$

Beachte: $z = a+ib$, $|e^{iz}| = |e^{ia-b}| = e^{-b} \leq 1 \quad b \geq 0$

$$= \mathrm{Re}\left[2\pi i\ \mathrm{Res}_i\left(\frac{e^{iz}}{(1+z^2)^2}\right)\right] = \mathrm{Re}\left[2\pi i\ \frac{\partial}{\partial z}\Big|_{z=i}\left(\frac{e^{iz}}{(i+z)^2}\right)\right]$$

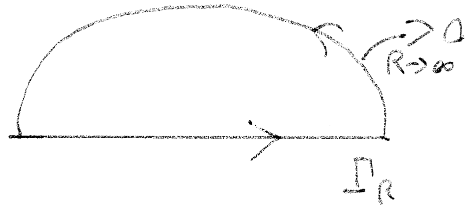
$$= \mathrm{Re}\left[2\pi i\left((-2)\frac{e^{iz}}{(2i)^3} + \frac{i e^{iz}}{(2i)^2}\right)\right]\Bigg|_{z=i}$$

$$= \mathrm{Re}\left[\frac{\pi}{2}\left(e^{-1} + e^{-1}\right)\right] = \underline{\frac{\pi}{e}}$$

(e) $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\,dx = \lim_{R\to\infty} \int_{\Gamma_R} \frac{z^2}{1+z^6}\,dz$$

$$= 2\pi i \left(\operatorname{Res}_{e^{i\pi/6}} + \operatorname{Res}_{e^{i\pi/2}} + \operatorname{Res}_{e^{i\frac{5}{6}\pi}} \right)$$

$$= \underline{\frac{\pi}{3}}$$



Mit $$\operatorname{Res}_{e^{i\pi/6}} \overset{1(b)}{=} \frac{(e^{i\pi/6})^2}{6(e^{i\pi/6})^5} = \frac{1}{6(e^{i\pi/6})^3} = \underline{\frac{1}{6i}}$$

$$\operatorname{Res}_{e^{i\pi/2}} = \frac{(e^{i\pi/2})^2}{6(e^{i\pi/2})^5} = \frac{1}{6e^{i\frac{3}{2}\pi}} = \underline{\frac{1}{6(-i)}}$$

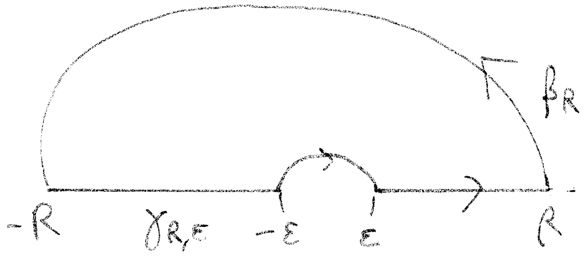
$$\operatorname{Res}_{e^{i\frac{5}{6}\pi}} = \frac{(e^{i\frac{5}{6}\pi})^2}{6(e^{i\frac{5}{6}\pi})^5} = \frac{1}{6e^{i\frac{15}{6}\pi}} = \underline{\frac{1}{6i}}$$

Oder substituieren mit $y = x^3$ und verwenden von

$$\int_{-\infty}^{\infty} \frac{1}{1+y^2}\,dy = \pi.$$

(f) Berechne $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx$

Betrachte die Funktion $g(z) = \dfrac{e^{iz}}{z}$ und die folgenden Wege:



$$\Gamma_{R,\varepsilon} = \gamma_{R,\varepsilon} + \beta_R$$

Dann gilt:
$$0 = \int_{\Gamma_{R,\varepsilon}} g(z)\,dz = \int_{\gamma_{R,\varepsilon}} g(z)\,dz + \int_{\beta_R} g(z)\,dz$$

Beachten nun:
$$\int_{\beta_R} g(z)\,dz = \int_0^\pi \frac{1}{Re^{it}} e^{i(e^{it}R)} Re^{it}\cdot i\,dt$$
$$= \int_0^\pi i e^{i(Re^{it})}\,dt = \underbrace{\int_0^\delta i e^{i(Re^{it})}\,dt}_{I_1} + \underbrace{\int_\delta^{\pi-\delta} i e^{i(Re^{it})}\,dt}_{I_2} + \underbrace{\int_{\pi-\delta}^{\pi} i e^{i(Re^{it})}\,dt}_{I_3}, \quad \delta > 0$$

$$|I_2| \leq \pi\cdot\max\left\{ \left|e^{i(Re^{it})}\right| \;\middle|\; t\in[\delta,\pi-\delta]\right\}$$
$$= \pi\cdot\max\left\{ \left|e^{-R\sin(t)}\right| \;\middle|\; t\in[\delta,\pi-\delta]\right\}$$
$$= \pi\cdot e^{-R\sin(\delta)} \xrightarrow{R\to\infty} \underline{0}$$

$$|I_1| \leq \delta\cdot\max\left\{ \left|e^{i(Re^{it})}\right| \;\middle|\; t\in[0,\delta]\right\} = \underline{\delta}$$

$$|I_3| \leq \underline{\delta} \quad \text{(analog)}$$

Insgesamt gilt also:

$$\lim_{R\to\infty} \left|\int_{\beta_R} g(z)\,dz\right| \leq 2\delta \quad \text{für alle } \delta > 0$$

$$\Rightarrow \underline{\int_{\beta_R} g(z)\,dz = 0}$$

Somit gilt: $\lim\limits_{R\to\infty} \int\limits_{\gamma_{R,\varepsilon}} g(z)\, dz = 0$

$$\Rightarrow \int\limits_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int\limits_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx = \left(\int\limits_{0}^{\pi} \frac{e^{i(\varepsilon e^{it})}}{\varepsilon e^{it}} \varepsilon i e^{it}\, dt \right)$$

$$= \int\limits_{\alpha_\varepsilon} \frac{e^{iz}}{z}\, dz \qquad \text{mit } \alpha_\varepsilon \text{ Halbkreis um } 0$$

$$= \frac{1}{2} \int\limits_{\partial B_\varepsilon(0)} \frac{e^{iz}}{z}\, dz + \mathcal{O}(\varepsilon) \qquad (\varepsilon \to 0^+)$$

$$= \frac{2\pi i}{2} \cdot \mathrm{Res}_0 \left(\frac{e^{iz}}{z} \right) + \mathcal{O}(\varepsilon) = \pi \cdot i \cdot e^{i \cdot 0} + \mathcal{O}(\varepsilon)$$

$$= \underline{\pi \cdot i + \mathcal{O}(\varepsilon)} \quad (*) \qquad (\varepsilon \to 0^+)$$

$$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \left(\int\limits_{-\infty}^{-\varepsilon} \mathrm{Im}\left(\frac{e^{ix}}{x}\right) dx + \int\limits_{\varepsilon}^{\infty} \mathrm{Im}\left(\frac{e^{ix}}{x}\right) dx \right)$$

$$= \mathrm{Im}\left[\lim_{\varepsilon \to 0^+} \left(\int\limits_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int\limits_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx \right) \right]$$

$$\overset{(*)}{=} \mathrm{Im}(\pi \cdot i) = \underline{\pi}$$

$$\Rightarrow \underline{\int\limits_{0}^{\infty} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}}$$