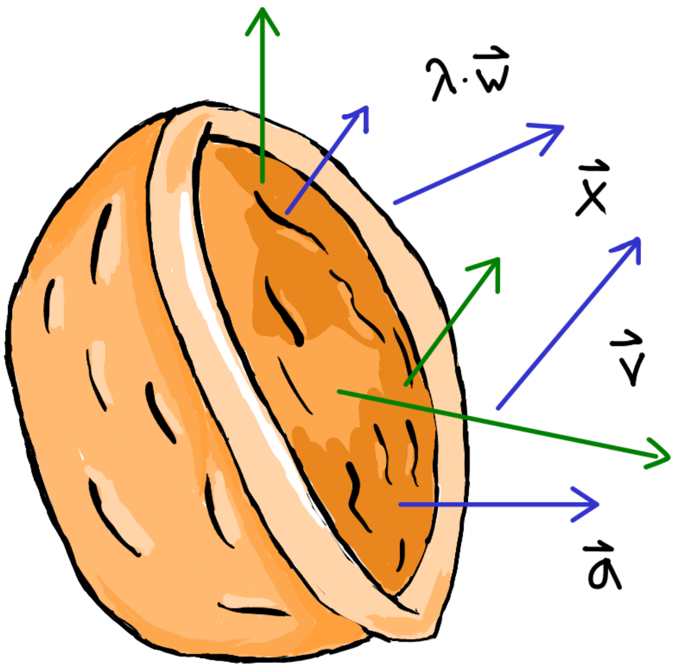
Linear Algebra

– in a nutshell –

This book was created and used for the lecture at Hamburg University of Technology in the winter term 2018/19 for General Engineering Science and Computer Science students.

Hamburg, 13th August 2022, Version 2.4



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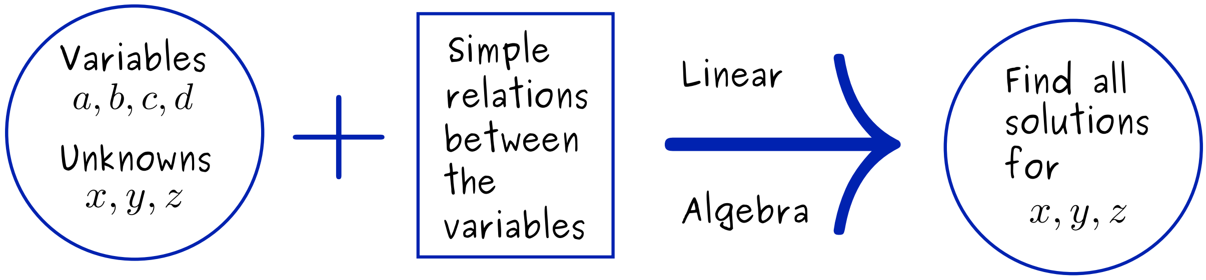
Contents

Some words

This text should help you to understand the course Linear Algebra. To expand your knowledge, you can look into the following books:

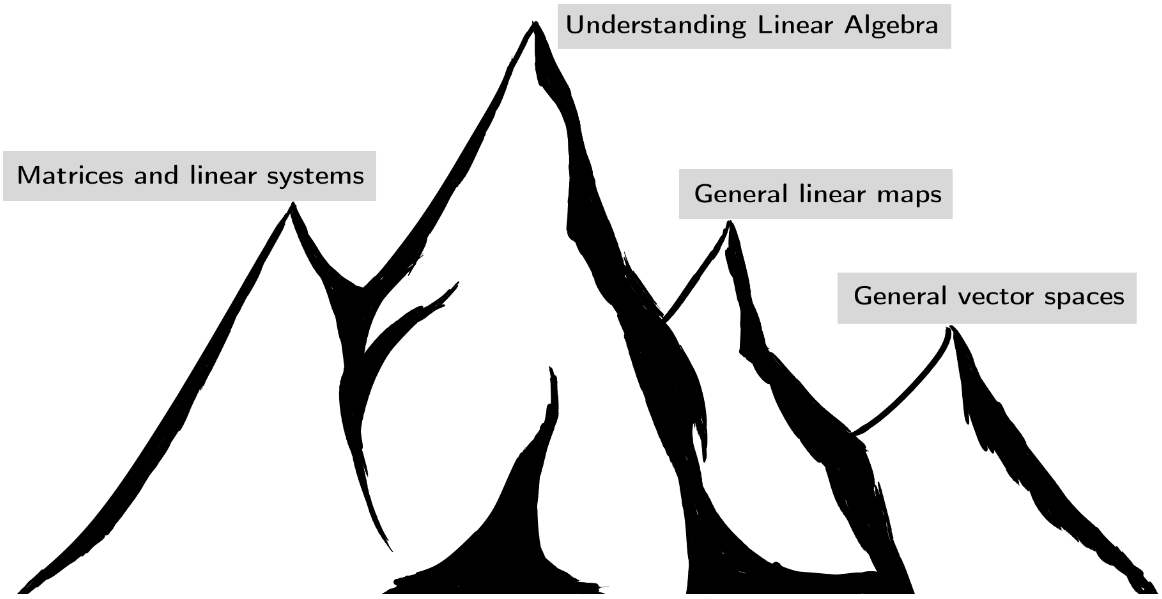
- Gilbert Strang: *Introduction to Linear Algebra*,
- Sheldon Axler: *Linear Algebra Done Right*,
- Gerald Teschl, Susanne Teschl: *Mathematik für Informatiker, Band 1*.
- Shin Takahashi, Iroha Inoue: *The Manga Guide to Linear Algebra*.
- Klaus Jänich: *Lineare Algebra*.

Linear Algebra is a very important topic and useful in different applications. We[1] discuss simple examples later. However, the main idea is that we have a problem consisting of a lot of quantities where some are fixed and others can be altered or are not known. However, if we know the relations between the quantities, we use *Linear Algebra* to find all the possible solutions for the *Unknowns*.



That would be the calculation side of the world regarding *Linear Algebra*. In this lecture, we will concentrate on understanding the field as a whole. Of course, this is not an easy task and it will be a hiking tour that we will do together. The summit and goal is to understand why solving equations is indeed a meaningful mathematical theory.

[1] In mathematical texts, usually, the first-person plural is used even if there is only one author. Most of the time it simply means "we" = "I (the author) and the reader".



We start in the valley of mathematics and will shortly scale the first hills. Always stay in shape, practise and don't hesitate to ask about the ways up. It is not an easy trip but you can do it. Maybe the following tips can guide you:

- You will need a lot of time for this course if you really want to *understand* everything you learn. Hence, make sure that you have enough time each week to do mathematics and keep these time slots clear of everything else.
- Work in groups, solve problems together and discuss your solutions. Learning mathematics is not a competition.
- Explain the content of the lectures to your fellow students. Only the things you can illustrate and explain to others are really understood by you.
- Learn the Greek letters that we use in mathematics:

α	alpha	β	beta	γ	gamma	Γ	Gamma
δ	delta	ϵ	epsilon	ε	epsilon	ζ	zeta
η	eta	θ	theta	Θ	Theta	ϑ	theta
ι	iota	κ	kappa	λ	lambda	Λ	Lambda
μ	mu	ν	nu	ξ	xi	Ξ	Xi
π	pi	Π	Pi	ρ	rho	σ	sigma
Σ	Sigma	τ	tau	υ	upsilon	Υ	Upsilon
ϕ	phi	Φ	Phi	φ	phi	χ	chi
ψ	psi	Ψ	Psi	ω	omega	Ω	Omega

  This video may help you there:
  https://jp-g.de/bsom/la/greek/

- Choosing a book is a matter of taste. Look into different ones and choose the book that really convinces you.
- Keep interested, fascinated and eager to learn. However, do not expect to understand everything at once.

DON'T PANIC J.P.G.

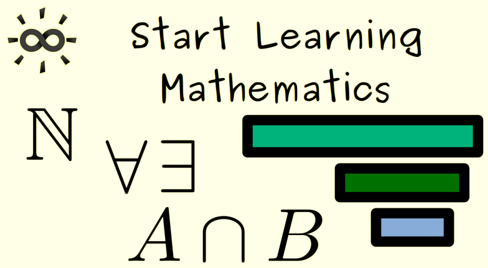
1 Foundations of mathematics

> It is a mistake to think you can solve any major problems just with potatoes.
>
> Douglas Adams

Before starting with *Linear Algebra*, we first have to learn the mathematical language, which consists of symbols, logic, sets, numbers, maps and so on. We also talk about the concept of a mathematical proof. These things build up the mathematical foundation.

A little bit of knowledge about numbers and how to calculate with them is assumed but not much more than that. All symbols are introduced such that you know how to work with them. However, if you interested in a more detailed discussion, I can recommend you my video series about the foundations of mathematics:

Video: Start Learning Mathematics



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1.1 Logic and sets

Basic logic is something, we usually accomplish intuitively right. However, in mathematics we have to define it in an unambiguous way and it may differ a little bit from the everyday logic. It is very important and useful to bring into our attention some of the basic rules and notations of logic. For Computer Science students, logic is considered in more detail in other courses.

Let us start with a definition:

Definition 1.1. logical statement, proposition

A logical statement (or proposition) is a statement, which means a meaningful declarative sentence, that is either true or false.

Instead of *true*, one often writes T or 1 and instead of *false*, one often writes F or 0.

Not every meaningful declarative fulfils this requirement. There are opinions, alternative facts, self-contradictory statements, undecidable statements and so on. In fact, a lot of examples here, outside the mathematical world, work only if we give the words unambiguous definitions which we will implicitly do.

Example 1.2. Which of these are logical statements?

(a) Hamburg is a city.

(b) $1 + 1 = 2$.

(c) The number 5 is smaller than the number 2.

(d) Good morning!

(e) $x + 1 = 1$.

(f) Today is Tuesday.

The last two examples are not logical statements but so-called predicates and will be considered later.

Logical operations

For given logical statements, one can form new logical statements with so-called *logical operations*. In the following, we will consider two logical statements A and B.

Definition 1.3. Negation $\neg A$ ("not A")

$\neg A$ is true if and only if A is false.

$$\text{Truth table} \qquad \begin{array}{c|c} A & \neg A \\ \hline T & F \\ F & T \end{array} \tag{1.1}$$

Example 1.4. What are the negations of the following logical statements?

(a) The wine bottle is full.

(b) The number 5 is smaller than the number 2.

(c) All students are in the lecture hall.

Definition 1.5. Conjunction $A \wedge B$ ("A and B")

$A \wedge B$ is true if and only if both A and B are true.