The Bright Side of Complex Analysis

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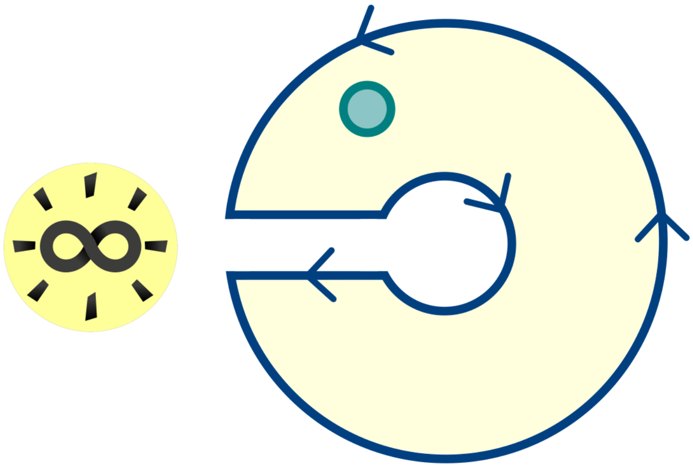
0.2.1.544

September 29, 2024



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Contents

Some words

This book was developed along my video series which is freely available on YouTube and can be used while reading this written form here. For this, you find QR codes at the beginning of each chapter of the book or clickable links in the digital version.

Overview for the whole course

Here you find all videos for the Complex Analysis Course.

Learning a whole mathematical topic like *Complex Analysis* can be seen as a whole hiking trip, which requires stamina and strength.

We start in the valley of mathematics and will shortly scale the first hills. Always stay in shape, practise and don't hesitate to ask about the ways up. It is not an easy trip but you can do it. Maybe the following tips can guide you:

- You will need a lot of time for this book if you really want to *understand* everything you learn. Hence, make sure that you have enough time each week to do mathematics and keep these time slots clear of everything else.
- Work in groups, solve problems together and discuss your solutions. Learning mathematics is not a competition.
- Explain the content of the lectures to your fellow students. Only the things you can illustrate and explain to others are really understood by you.
- Learn the Greek letters that we use in mathematics:

α	alpha	β	beta	γ	gamma	Γ	Gamma
δ	delta	ϵ	epsilon	ε	epsilon	ζ	zeta
η	eta	θ	theta	Θ	Theta	ϑ	theta
ι	iota	κ	kappa	λ	lambda	Λ	Lambda
μ	mu	ν	nu	ξ	xi	Ξ	Xi
π	pi	Π	Pi	ρ	rho	σ	sigma
Σ	Sigma	τ	tau	υ	upsilon	Υ	Upsilon
ϕ	phi	Φ	Phi	φ	phi	χ	chi
ψ	psi	Ψ	Psi	ω	omega	Ω	Omega

A quick overview for Greek letters

Watch Greek letters in YouTube.

- Choosing a book is a matter of taste. Look into different ones and choose the book that really convinces you.
- Keep interested, fascinated and eager to learn. However, do not expect to understand everything at once.
- If you are unsure if you have all the prerequisites to understand this series about Complex Analysis, check my website to see the network for your learning path: https://thebrightsideofmathematics.com/startpage/. Also do this whenever a used notion seems too unfamiliar to you.

DON'T PANIC

1 Introduction

Complex Analysis is nice and also fun. We will have a lot of videos here and you find them at the beginning of each chapter.

Introduction to Complex Analysis

Watch the introduction on YouTube.

The numbering of the chapters coincides with the numbering of the videos and all definitions and theorems are numbered with a leading chapter number. This should help you to navigate this book and the additional video material.

In short, we can say that *Complex Analysis* is the analysis of differentiable functions $f : \mathbb{C} \to \mathbb{C}$. This is in contrast to *Real Analysis*, where only functions $f : \mathbb{R} \to \mathbb{R}$ are considered. It turns out that increasing the number set in such a way adds a lot of different properties to the differentiable functions. We will also see that even real-valued problems can be solved with this detour over the complex numbers.

Example 1.1. Improper Riemann integrals

The following integral might be hard to solve with methods from real analysis:

$$\int_{\infty}^{\infty} \frac{x \sin(x)}{1 + x^2} \, dx \, .$$

However, at the end of the course, we can easily conclude that the value is $\frac{\pi}{e}$.

You do not need a lot of prior knowledge to understand most of this course, but you definitely need to know the *complex numbers* as we have them introduced in the *Start Learning Mathematics* course, which you can also find on my webpage. On the other hand, we will talk about derivatives, power series, and integrals in the complex realm such that some knowledge about these topics from real analysis is helpful but not extremely necessary since we will explain a lot anyway. Just make sure that you don't get lost and always check out some videos from the *Real Analysis* series if needed.

So let's start with the basic definitions we need:

Definition 1.2. Distance function

The set of the complex numbers $\mathbb{C}$ *becomes a so-called* metric space *with the distance* *function*

$$d(z, w) := |z - w|$$

where we just have the absolute value in complex numbers.

Definition 1.3. Convergence

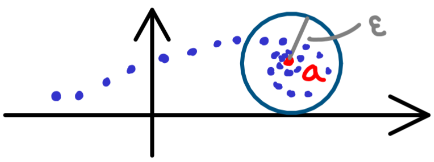
A sequence of complex numbers, denoted by $(z_n)_{n\in\mathbb{N}}$*, is called* convergent *to a complex number* $a \in \mathbb{C}$ *if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ : \ |z_n - a| < \varepsilon\,.$$

In this case, the number a *is called the* limit *of the sequence* $(z_n)_{n\in\mathbb{N}}$ *and we write:*

$$z_n \xrightarrow{n\to\infty} a$$

We say that the sequence lies in each small ε-ball around a, eventually.



So no matter how small the ball is chosen, eventually, all infinitely many remaining sequence members lie in it. We say *ball* in the general meaning, but in this picture, it is a two-dimensional disc. However, we keep the general name:

Definition 1.4. Epsilon-Ball

For a real number $\varepsilon > 0$ *and complex number* $a \in \mathbb{C}$*, we define the* ε-ball *as*

$$B_\varepsilon(a) := \{z \in \mathbb{C} \mid |z - a| < \varepsilon\}$$

Please note that convergence of a complex sequence $(z_n)_{n\in\mathbb{N}}$ with limit a is equivalent to saying that the real sequence

$$(|z_n - a|)_{n\in\mathbb{N}}$$

is a convergent in the real numbers with limit 0.

Since we can measure distances between complex numbers as we can do it for real numbers, all notions that just need a distance immediately generalize from real functions to complex functions. For example, we know that continuous functions send close points to close points again. This means we can directly define such functions:

Definition 1.5. Continuity

A function $f : \mathbb{C} \to \mathbb{C}$ *is called* continuous at $z_0 \in \mathbb{C}$ *if, for all sequence* $(z_n)_{n\in\mathbb{N}}$ *in the complex numbers, we have:*

$$z_n \xrightarrow{n\to\infty} z_0 \quad \Longrightarrow \quad f(z_n) \xrightarrow{n\to\infty} f(z_0)\,.$$