Lösung:

Zuerst: $t(x) = \tan\left(\frac{x}{2}\right) = t$

Dann: $\cos^2\left(\frac{x}{2}\right) = \dfrac{\sin^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)} \Rightarrow \cos^2\left(\frac{x}{2}\right) = \dfrac{1}{t^2+1}$

und somit: (mit Additionstheoremen)

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)$$

$$= \frac{2t}{t^2+1}$$

$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$= 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{1-t^2}{t^2+1}$$

$$\frac{dt}{dx} = \frac{1}{2}\,\frac{1}{\cos^2\left(\frac{x}{2}\right)} = \frac{1+t^2}{2}$$

Dann ergeben die Integrale:

(a) $$\int \frac{1}{\sin(x)+\cos(x)}\,dx = \int \frac{2}{1+2t-t^2}\,dt \overset{\text{Partialbruchzerlegung}}{=}$$

$$= \int \left(\frac{1/\sqrt{2}}{t+(1+\sqrt{2})} - \frac{1/\sqrt{2}}{t-(1-\sqrt{2})} \right) dt$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{\tan\left(\frac{x}{2}\right) - (1+\sqrt{2})}{\tan\left(\frac{x}{2}\right) - (1-\sqrt{2})} \right| + C, \quad C \in \mathbb{R}.$$

(b)

$$\int \frac{\tan(x)}{1+\sin(x)}\,dx = \int \frac{2t \cdot 2}{(1-t^2)\left(1+\frac{2t}{1+t^2}\right)(1+t^2)}\,dt$$

$$= \int \frac{4t}{(1-t)(1+t)^3}$$ Partialbruchzerlegung

$$= \int \left(\frac{1/2}{1-t} + \frac{1/2}{1+t} + \frac{1}{(1+t)^2} - \frac{2}{(1+t)^3} \right) dt$$

$$= -\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t| - \frac{1}{1+t} + \frac{1}{(1+t)^2} + C_1$$

$C_1 \in \mathbb{R}$